

MULTIDIMENSIONAL LOCALIZED SOLITONS *

M. Boiti L. Martina
F. Pempinelli

Dipartimento di Fisica dell'Università e Sezione INFN[†]
73100 Lecce, ITALIA

August 25, 1993 (to be published in *Chaos Solitons & Fractals*)

Abstract

Recently it has been discovered that some nonlinear evolution equations in $2+1$ dimensions, which are integrable by the use of the Spectral Transform, admit localized (in the space) soliton solutions. This article briefly reviews some of the main results obtained in the last five years thanks to the renewed interest in soliton theory due to this discovery. The theoretical tools needed to understand the unexpected richness of behaviour of multidimensional localized solitons during their mutual scattering are furnished. Analogies and especially discrepancies with the unidimensional case are stressed.

1 Introduction

1.1 Special features of solitons in two dimensions

Since the discovery of the soliton in 1965 by Zabusky and Kruskal a large new domain of mathematical physics developed and is believed to have reached maturity. It is generically called the soliton theory. Its principal mathematical tool is the so called Spectral (or Scattering) Transform that is used to solve a large class of nonlinear evolution equations in $1+1$ (one spatial and one temporal) dimensions.

Some of these equations, in particular the nonlinear Schrödinger (NLS) equation and its generalizations, can be obtained in an appropriate multiscale limit from a very large class of nonlinear dispersive equations. Therefore, it is not astonishing that applications of soliton theory are percolating through the whole of physics, especially quantum field theory, solid state physics, nonlinear optics, plasma physics and hydrodynamics, and other natural sciences.

The most impressive phenomenon in the theory and in the applications is the existence of solitons, i.e. (localized) coherent structures that mutually interact preserving their individuality.

In the last decade many efforts have been made to extend the soliton theory to nonlinear evolution equations in $2+1$ (two spatial and one temporal) dimensions. In fact

*Work supported in part by Ministero delle Università e della Ricerca Scientifica e Tecnologica, Italy

[†]e-mail: BOITI@LECCE.INFN.IT, MARTINA@LECCE.INFN.IT and PEMPI@LECCE.INFN.IT

the Spectral Transform was extended to dispersive nonlinear evolution equations in 2+1 dimensions but it was generally admitted the lack of two dimensional localized solitons. Only recently, in 1988, it has been discovered (by Boiti, Leon, Martina and Pempinelli) that all the equations in the hierarchy related to the Zakharov–Shabat hyperbolic spectral problem in the plane have (exponentially) localized soliton solutions. The most representative equation in the hierarchy is the Davey–Stewartson I equation (DSI), which provides a two dimensional generalization of the NLS equation.

This discovery has stimulated a renewed interest in soliton theory. The first results are very promising. In particular soliton solutions display a richer phenomenology than in 1+1 dimensions. This opens the way to applications in multidimensions, which, hopefully, are expected to be even more interesting than in one space dimension.

In contrast with the 1+1 dimensional case the time evolution of the solution of the Davey–Stewartson equation is not uniquely determined by the initial data. In addition one has to give, at all times, boundary data. If they are chosen to be identically zero solitons cannot be present. For a convenient choice the solution can contain solitons but not necessarily does. In the affirmative case the boundary data fix the kinematics of the incoming and outgoing scattering solitons, i.e. their velocities and locations in the plane in the far past and in the far future. The initial data fix the dynamics of the interaction.

The scattering of the solitons can be inelastic and they can change shape and also exchange mass (energy or charge according to the specific physical interpretation). In fact, while the total mass of solitons is conserved, the mass of the single soliton, in general, is not preserved by the interaction and solitons can also simulate inelastic scattering processes of quantum particles as creation and annihilation, fusion and fission, and interaction with virtual particles.

1.2 Guidelines for additional reading

One main feature of the nonlinear dynamical systems is that different non equivalent approaches are possible and each one is useful and clarifying from a special point of view.

Also multidimensional solitons can and, effectively, have been studied by using different tools. In this article, mainly for lack of space, we made the (questionable) choice of using only the Bäcklund transformations and a special version of the Spectral Transform. In fact, we collected, reorganized and simplified the results that the reader can find scattered, with some minor additional details, in references [1]–[8]. In the first article of the list the localized solitons in the plane have been discovered and in the second one a preliminary analysis of the properties of the new Spectral Transform needed to describe them is performed. These two papers opened the way to a deeper understanding of integrable nonlinear evolution equations in multidimensions and to the search and discovery of other dynamical systems admitting localized coherent structures.

Another special version of the Spectral Transform theory different from that one presented in this paper has been developed in [9]–[11]. This alternative theory is relevant because it has been used to clarify the role played by the boundaries and to show that multidimensional solitons, in contrast with the one dimensional solitons, can interact inelastically. These authors suggested to call these solitons dromions in order to stress the fact that they can be driven everywhere in the plane along tracks (dromos in greek) by choosing a suitable motion of the boundaries.

In both versions the Spectral Transform for the nonstationary Schrödinger equation (with a potential vanishing at large distances in every direction except a finite number) plays a fundamental role. To extend the theory, originally developed for potentials vanishing at large distances [12], to this case it has been necessary to introduce a new mathematical entity resembling the resolvent of the linear operator theory. For lack of space this topic has been skipped in this article. The interested reader can consult [13, 14]. The multi-soliton solution of the Kadomtsev–Petviashvili I (KPI) equation, that is needed in order to build the multi-soliton solution of the DSI and DSIII systems, has been derived in [15] and in its most general form in [16].

Also other methods have been used to get the multidimensional solitons, the bilinear approach [17], the quantum machinery of creation and annihilation operators [18, 19], the Grammian determinants [20], the Darboux transformations [21] and the dressing method [22]. The most general form of the multi-soliton solution together with the remark that the number of solitons is not necessarily conserved were first given in [17].

The search of multidimensional solitons has been extended by different authors to other nonlinear evolution equations in $2 + 1$ dimensions [23]–[26]. Some interesting attempts using the $\bar{\partial}$ -method and a direct method have been done also in higher dimensions, precisely in [27] and [28]–[30].

In [31]–[35] it has been shown that the DSI and, successively in [36], that also the DSIII system can be obtained by using a multiscale limit starting from a very large class of dispersive nonlinear equations. In particular the DSI equation with boundaries compatible with solitons can be embedded into the KPI equation [37]. These results open the way to the search of physical applications other than hydrodynamics, studied in [38, 39, and references quoted therein]. In this respect a more discouraging analysis is made in [40]. This last paper, mainly dedicated to the consequences of the lack of conserved quantities for DSI, contains also an interesting examination of the literature dealing with the discovery of the DSI system and with the early attempts to build the corresponding Spectral Transform.

For the Hamiltonian version of the DSI system and its quantum extension see [41]–[45]. A Hamiltonian version of the DSIII system appears already in [46] and a bi-Hamiltonian version in [47]. For the general method for building integrable nonlinear evolution equations starting from a $\bar{\partial}$ -problem that allowed to prove that DSIII is integrable see [48]–[50] and, specifically, [51]. For more details on the localized soliton solutions of DSIII see [52]. For the DSII system which is related to the elliptic version of the Zakharov–Shabat problem in the plane see [53, 54] for the Spectral Transform theory and [55, 56] for the singular soliton solutions.

Finally, between the different excellent existing books on solitons let us suggest to the reader [57] and [58], as the most accurate and complete for the nonlinear evolution equations associated to the Schrödinger and to the Zakharov–Shabat spectral equation in $1+1$ dimensions, respectively, and [59] as the most updated and comprehensive. In particular for those who want to have a general overlooking on the subject and a rich bibliography to pick over this book is particularly recommended. A new book on multidimensional soliton has been just announced [60]. For the Bäcklund and Darboux transformations see [61] and [62]. Special attention to the algebraic and geometric approach to the soliton equations has been dedicated in [63] and to the multidimensional case in [64].

In all these books the reader can find many useful references for the problems considered in this paper and for related problems. Here we want to quote only some references

more relevant historically or more close to our specific approach. Precisely [65, 66] for the discovery of the one dimensional solitons, [67, 68] for the extension of the theory to the nonlinear Schrödinger equation, [69]–[71] for the introduction of the so-called $\bar{\partial}$ -method, [72]–[74], [54], [12], [75]–[80] for the extension of the Spectral Transform to 2 + 1 dimensions, [76]–[79] in particular for the introduction of the “weak” Lax representation of the integrable equations in 2 + 1 dimensions, [81, 82] for the Bäcklund transformations and [83] for the extension of the Bäcklund transformations to 2 + 1 dimensions.

By offering these guidelines for recovering interesting references we hope to redress the omissions due to oversight that have certainly occurred. Anyway, we apologize for this to the reader as well to the authors who may have unjustifiably excluded.

1.3 The Davey–Stewartson I equation

The Davey–Stewartson (DS) systems model the evolution of weakly nonlinear water waves that travel predominantly in one direction, are nearly monochromatic and are slowly modulated in the two horizontal directions. We are interested in the special DS system (DSI equation) that one gets in the shallow water limit when the effects of the surface tension are important. In characteristic coordinates and dimensionless form the DSI equation is a system of two coupled equations

$$\begin{aligned} iq_t + q_{uu} + q_{vv} - (\varphi_u + \varphi_v - \sigma_0 |q|^2)q &= 0 \\ 2\varphi_{uv} &= \sigma_0(|q|^2)_u + \sigma_0(|q|^2)_v, \quad \sigma_0 = \pm 1 \end{aligned} \quad (1.1)$$

where $q(u, v, t)$ is the (complex) envelope of the free surface of the water wave we are considering and $\varphi(u, v, t)$ is the (real) velocity potential of the mean motion generated by the surface wave.

It is worth to stress that the DSI system is not necessarily placed in the context of water waves. Indeed, it has been shown that a very large class of nonlinear dispersive equations in 2+1 dimensions reduces in an appropriate asymptotic limit to the DSI equation and therefore we expect it to arise in many different physical applications.

To exhibit explicitly the boundary value of φ at large distance in the (u, v) plane allowed by the second equation in (1.1) it is convenient to introduce the two fields

$$\begin{aligned} A^{(1)} &= -\varphi_v + \frac{1}{2}\sigma_0|q|^2 \\ A^{(2)} &= \varphi_u - \frac{1}{2}\sigma_0|q|^2 \end{aligned} \quad (1.2)$$

and to rewrite the DSI equation as

$$iq_t + q_{uu} + q_{vv} + (A^{(1)} - A^{(2)})q = 0 \quad (1.3)$$

$$\begin{aligned} A^{(1)} &= -\frac{1}{2} \int_{-\infty}^u du' \sigma_0(|q|^2)_v + a_0^{(1)}(v, t) \\ A^{(2)} &= \frac{1}{2} \int_{-\infty}^v dv' \sigma_0(|q|^2)_u + a_0^{(2)}(u, t) \end{aligned}$$

where $a_0^{(1)}$ and $a_0^{(2)}$ are the arbitrary boundaries.

In fact we could make another not equivalent choice and write the DSI equation as

$$iq_t + q_{uu} + q_{vv} + (A^{(1)} - A^{(2)})q = 0 \quad (1.4)$$

$$\begin{aligned} A^{(1)} &= -\frac{1}{4} \left(\int_{-\infty}^u + \int_{+\infty}^u \right) du' \sigma_0(|q|^2)_v + A_0^{(1)}(v, t) \\ A^{(2)} &= \frac{1}{4} \left(\int_{-\infty}^v + \int_{+\infty}^v \right) dv' \sigma_0(|q|^2)_u + A_0^{(2)}(u, t) \end{aligned}$$

where now $A_0^{(1)}(v, t)$ and $A_0^{(2)}(u, t)$ are the arbitrary boundaries.

It can be shown that the proper boundary conditions to be chosen are dictated by the specific multiscale limit one is choosing in getting the DSI equation. Specifically, equation (1.3) can be obtained via a multiscale limit from the Kadomtsev–Petviashvili (KPI) evolution equation while maintaining well posedness in time.

In order to solve the DSI equation it is convenient to introduce its more general two component version

$$iQ_t + \sigma_3(Q_{uu} + Q_{vv}) + [A, Q] = 0 \quad (1.5)$$

where

$$Q = \begin{pmatrix} 0 & q(u, v, t) \\ r(u, v, t) & 0 \end{pmatrix} \quad (1.6)$$

and

$$A = \begin{pmatrix} A^{(1)} & 0 \\ 0 & A^{(2)} \end{pmatrix} \quad (1.7)$$

with

$$\begin{aligned} A^{(1)} &= -\frac{1}{2} \int_{-\infty}^u du' (Q^2)_v + a_0^{(1)}(v, t) \\ A^{(2)} &= \frac{1}{2} \int_{-\infty}^v dv' (Q^2)_u + a_0^{(2)}(u, t) \end{aligned} \quad (1.8)$$

or

$$\begin{aligned} A^{(1)} &= -\frac{1}{4} \left(\int_{-\infty}^u + \int_{+\infty}^u \right) du' (Q^2)_v + A_0^{(1)}(v, t) \\ A^{(2)} &= \frac{1}{4} \left(\int_{-\infty}^v + \int_{+\infty}^v \right) dv' (Q^2)_u + A_0^{(2)}(u, t) \end{aligned} \quad (1.9)$$

according to the boundary conditions we choose. The previous considered equations are simply obtained for

$$r = \sigma_0 \bar{q} \quad (1.10)$$

(where \bar{q} denotes the complex conjugate of q) and are called reduced DSI equations in contrast with (1.5), (1.8) and (1.9) which can be named DSI equations with no additional specification.

The DSI equation can be obtained as the compatibility condition (Lax representation)

$$[T_1, T_2] = 0 \quad (1.11)$$

for two underlying linear operators

$$T_1(Q) = \partial_x + \sigma_3 \partial_y + Q \quad (1.12)$$

$$T_2(Q) = i\partial_t + \sigma_3 \partial_y^2 + Q \partial_y - \frac{1}{2} \sigma_3 Q_x + \frac{1}{2} Q_y + A \quad (1.13)$$

where

$$x = \frac{1}{2}(u + v), \quad y = \frac{1}{2}(u - v). \quad (1.14)$$

and σ_3 is the Pauli matrix.

The first of these operators is the Zakharov–Shabat hyperbolic spectral operator in the plane and can be considered to define a linear spectral problem in which Q plays the role of the data. In the case (1.8) with $a_0^{(1)}$ and $a_0^{(2)}$ considered as boundaries given at all times one can define the Spectral Transform of Q and solve the initial value problem for the DSI. This is explicitly done in section 3.

In the case (1.9) one can introduce the Hamiltonian

$$H = \iint dudv \left[r(\partial_u^2 + \partial_v^2)q - \frac{1}{4}qr(\partial_u \partial_v^{-1} + \partial_v \partial_u^{-1})qr + (A_0^{(1)} - A_0^{(2)})qr \right] \quad (1.15)$$

and the canonical Poisson brackets

$$\{F, G\} = i \iint dudv \left[\frac{\delta F}{\delta q} \frac{\delta G}{\delta r} - \frac{\delta F}{\delta r} \frac{\delta G}{\delta q} \right] \quad (1.16)$$

where q and r are the conjugate variables. Then the equations of motion

$$q_t = \{q, H\}, \quad r_t = \{r, H\} \quad (1.17)$$

yield the DSI equation. The problem of defining a Spectral Transform is completely open and, moreover, only in the special case $A_0^{(1)} \equiv A_0^{(2)} \equiv 0$ it has been shown that the DSI equation is Hamiltonian with a continuous infinity of independent commuting constants and is completely integrable in the Hamiltonian sense. This case is usually named Hamiltonian case.

1.4 The Davey–Stewartson III equation

There is another nonlinear evolution equation that can be associated to the Zakharov–Shabat hyperbolic spectral operator in the plane and that admits localized soliton solutions.

To get it we need to introduce a weaker form of the Lax representation (1.11). Precisely, we search for a second spectral operator T_2 that commute with the Zakharov–Shabat spectral operator T_1 in (1.12) only on the subspace of the eigenfunctions of T_1 (“weak” Lax representation)

$$T_1 \psi = 0, \quad [T_1, T_2] \psi = 0. \quad (1.18)$$

For

$$T_1 = 2 \operatorname{diag}(\partial_u, \partial_v) + Q \quad (1.19)$$

$$T_2 = i\partial_t + \partial_u^2 + \partial_v^2 + A + \begin{pmatrix} 0 & q_u \\ r_v & 0 \end{pmatrix} \quad (1.20)$$

we have the so-called DSIII system

$$iQ_t + \sigma_3 (Q_{vv} - Q_{uu}) + [A, Q] = 0 \quad (1.21)$$

$$A_u^{(1)} = -\frac{1}{2}(Q^2)_v \quad (1.22)$$

$$A_v^{(2)} = -\frac{1}{2}(Q^2)_u \quad (1.23)$$

to be compared with the DSI system in (1.5) and (1.8) or (1.9). The DSIII system, as DSI, is compatible with the reduction (1.10).

Of course also DSI can be obtained by using a “weak” Lax representation instead of the usual “strong” one in (1.11). Then, one can choose for T_2 instead of (1.13)

$$T_2 = i\partial_t + \partial_v^2 - \partial_u^2 + A + \begin{pmatrix} 0 & -q_u \\ r_v & 0 \end{pmatrix} \quad (1.24)$$

and get again DSI. The theory of the Bäcklund transformation and of the Spectral Transform developed in the following sections does not change if the starting Lax pair is “weak” or “strong”. Therefore, comparison of definition (1.20) with (1.24) makes clear that one cannot expect any difference, apart some signs, between formulae for DSI and DSIII.

In particular the time evolution of ψ for DSIII has to be fixed as follows

$$T_2\psi = -k^2\psi \quad (1.25)$$

to be compared with (3.62) for DSI.

The DSIII system admits for convenient boundaries of the form

$$A^{(1)} = -\frac{1}{2} \int_{-\infty}^u du' (Q^2)_v + a_0^{(1)}(v, t) \quad (1.26)$$

$$A^{(2)} = -\frac{1}{2} \int_{-\infty}^v dv' (Q^2)_u + a_0^{(2)}(u, t)$$

localized solitons of the same shape of those of DSI, which exhibit similar dynamical phenomena but evolve differently in time.

We write here the one soliton solution

$$q = -\frac{2\lambda_{\Im}\eta \exp[i\varphi]}{D}, \quad r = -\frac{2\mu_{\Im}\rho \exp[-i\varphi]}{D} \quad (1.27)$$

where

$$\varphi = \mu_{\Re}u + \lambda_{\Re}v + (\lambda_{\Im}^2 - \lambda_{\Re}^2 - \mu_{\Im}^2 + \mu_{\Re}^2)t, \quad (1.28)$$

$$D = 2\gamma(\cosh \xi_1 + \cosh \xi_2) + \exp(\xi_2), \quad \gamma = \frac{1}{4}\eta\rho, \quad (1.29)$$

$$\xi_1 = -\mu_{\Im}u - \lambda_{\Im}v + 2(\lambda_{\Im}\lambda_{\Re} - \mu_{\Im}\mu_{\Re})t, \quad (1.30)$$

$$\xi_2 = \mu_{\Im}u - \lambda_{\Im}v + 2(\lambda_{\Im}\lambda_{\Re} + \mu_{\Im}\mu_{\Re})t \quad (1.31)$$

to be compared with (2.40)–(2.44). As in the DSI case the complex parameters $\lambda = \lambda_{\Re} + i\lambda_{\Im}$ and $\mu = \mu_{\Re} + i\mu_{\Im}$ are the discrete eigenvalues of the associated Zakharov–Shabat spectral problem and ρ and η are arbitrary complex constants satisfying the conditions $\gamma \in \mathbb{R}$ and $\gamma(1 + \gamma) > 0$. The multi-soliton solution can be easily obtained by taking the same lines we choose in the following sections for the DSI equation.

2 Solitons via Bäcklund Transformations

The Bäcklund transformations have their origin in work by Bäcklund in the late nineteenth century and are, therefore, the oldest tool used in exploring nonlinear integrable systems. Much more recently many different sophisticated and powerful methods have been developed, in particular the Spectral Transform and the dressing method. However, in our opinion, the Bäcklund transformations remain the simplest way for getting the soliton solutions. Moreover, because, under appropriate circumstances, a reiterated application of the Bäcklund transformations generate a sequence of solutions by a purely algebraic superposition principle they can be used to study the interaction properties of the solitons.

The simplest way to generate a Bäcklund transformation is to use the gauge invariance of the linear spectral problem associated to the nonlinear evolution equation one is considering. The gauge that generates the Bäcklund transformation is called Bäcklund gauge. The localized soliton solutions of the DSI equation, with boundaries of the form in (1.8), were for the first time derived by using these special gauge transformations. Successively they have been rederived by using the techniques of the Spectral Transform. But, in the case of the Hamiltonian DSI equation, we have not, presently, at our disposal the Spectral Transform or the dressing method and, consequently, in order to get explicit solutions we are left with the necessity to generalize the Bäcklund gauges such as to include also the special form of the boundaries in (1.9). To choosing these boundaries corresponds to imposing to the solutions nonlinear constraints, which can be solved only by using the additional freedom at our disposal in the generalized Bäcklund gauges. We are able to write explicitly infinite wave solutions with constant and periodically modulated amplitudes.

2.1 Generalized Bäcklund gauge transformation

Once given a solution Q of the DSI equation we want to generate a new solution Q' of the same equation by using a convenient gauge operator B that transforms according to the equation

$$\psi' = B(Q', Q)\psi \quad (2.1)$$

the matrix solution ψ of the principal spectral equation

$$T_1(Q)\psi = 0 \quad (2.2)$$

for Q to the matrix solution ψ' of the same spectral problem

$$T_1(Q')\psi' = 0 \quad (2.3)$$

for Q' .

It is easy to verify that if B satisfies

$$T_1(Q')B(Q', Q) - B(Q', Q)T_1(Q) = 0 \quad (2.4)$$

$$T_2(Q')B(Q', Q) - B(Q', Q)T_2(Q) = 0 \quad (2.5)$$

then $T_1(Q')$ and $T_2(Q')$ satisfy the same compatibility condition

$$[T_1, T_2] = 0 \quad (2.6)$$

as $T_1(Q)$ and $T_2(Q)$ and, consequently, Q' satisfies the DSI equation. In this case the gauge B is called Bäcklund gauge and the equations (2.4) and (2.5) yield, respectively, the so-called space and time component of the Bäcklund transformation.

Non trivial Bäcklund gauges are polynomial in the operator ∂_y . We are interested in the most general Bäcklund gauge of first order of the form

$$B(Q', Q) = \alpha \partial_y + B_0(Q', Q) \quad (2.7)$$

with α a constant diagonal matrix and B_0 a matrix. By inserting it in (2.4) we get its functional form

$$B(Q', Q) = \alpha \partial_y - \frac{1}{2} \sigma_3 (Q' \alpha - \alpha Q) - \frac{1}{2} \sigma_3 \alpha \mathcal{I}(Q'^2 - Q^2) + \beta \quad (2.8)$$

and the space component of the Bäcklund transformation

$$\begin{aligned} Q' \left[\beta - \frac{1}{2} \alpha \sigma_3 \mathcal{I}(Q'^2 - Q^2) \right] - \left[\beta - \frac{1}{2} \alpha \sigma_3 \mathcal{I}(Q'^2 - Q^2) \right] Q \\ - \frac{1}{2} \sigma_3 (Q' \alpha - \alpha Q)_x - \frac{1}{2} (Q' \alpha + \alpha Q)_y = 0. \end{aligned} \quad (2.9)$$

The matrix operator \mathcal{I} is defined by

$$\mathcal{I} = (\partial_x + \sigma_3 \partial_y)^{-1} \quad (2.10)$$

and the diagonal matrix β is subjected to the constraint

$$(\partial_x + \sigma_3 \partial_y) \beta = 0, \quad (2.11)$$

i.e. it is of the form

$$\beta = \begin{pmatrix} \beta_1(v, t) & 0 \\ 0 & \beta_2(u, t) \end{pmatrix} \quad (2.12)$$

where

$$u = x + y, \quad v = x - y \quad (2.13)$$

and β_1 and β_2 are arbitrary functions. Note that, in contrast with the 1+1 dimensional case, the matrix β that plays the role of ‘constant of integration’ in the solution of (2.4) admits also a space dependence. This additional freedom will be used in the following for getting soliton solutions of the Hamiltonian DSI equation.

By inserting $B(Q', Q)$ in (2.5) we get the time component of the Bäcklund transformation, which can be shown by use of (2.9) to be equivalent to the DSI for Q' , and two additional partial differential equations

$$\left[\beta - \frac{1}{2} \alpha \sigma_3 \mathcal{I}(Q'^2 - Q^2) \right]_y + \frac{1}{2} \alpha \sigma_3 (A' - A) + \frac{1}{4} \alpha (Q'^2 - Q^2) = 0 \quad (2.14)$$

$$\begin{aligned} i \left[\beta - \frac{1}{2} \alpha \sigma_3 \mathcal{I}(Q'^2 - Q^2) \right]_t + (A' - A) \left[\beta - \frac{1}{2} \alpha \sigma_3 \mathcal{I}(Q'^2 - Q^2) \right] \\ - \frac{1}{2} \alpha (A' + A)_y + \frac{1}{4} \alpha \sigma_3 (Q'^2 + Q^2)_y - \frac{1}{4} [(\partial_x + \sigma_3 \partial_y) Q'] Q' \alpha \\ - \frac{1}{4} \alpha Q [(\partial_x - \sigma_3 \partial_y) Q] + \frac{1}{4} (\partial_x - \sigma_3 \partial_y) (Q' \alpha Q) = 0. \end{aligned} \quad (2.15)$$

These two equations can be used for determining the field A' and the admissible β 's. It can be verified that they are compatible with the equation (2.11) for β by applying the

operator $(\partial_x + \sigma_3 \partial_y)$ to both of them and by showing that the two obtained equations are identically satisfied for Q and Q' solutions of the DSI equation and of the space component of the Bäcklund transformation.

The Bäcklund gauge $B(Q', Q)$ for general α and β in (2.8) can be obtained by composing two simpler Bäcklund gauges that are called elementary Bäcklund gauges of the first and second kind. They are obtained by choosing, respectively,

$$\alpha_I = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta_I = \begin{pmatrix} \lambda(v, t) & 0 \\ 0 & 1 \end{pmatrix} \quad (2.16)$$

and

$$\alpha_{II} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \beta_{II} = \begin{pmatrix} 1 & 0 \\ 0 & \mu(u, t) \end{pmatrix}, \quad (2.17)$$

and are noted $B_I(Q', Q; \lambda)$ and $B_{II}(Q', Q; \mu)$. They are not compatible with the reduction

$$r = \sigma_0 \bar{q} \quad (2.18)$$

but by composing two elementary Bäcklund gauges of different kind one can get finally a solution satisfying the reduction.

This procedure simplifies radically the computation because, according to the general feature of the Bäcklund transformations, the recursive application of the Bäcklund transformations can be achieved by purely algebraic means. An additional simplification is obtained by imposing the commutativity of the diagram

$$\begin{array}{ccc} & Q_I & \\ B_I(\lambda) \nearrow & & \searrow B_{II}(\mu) \\ Q & & Q' \\ B_{II}(\mu) \searrow & & \nearrow B_I(\lambda) \\ & Q_{II} & \end{array}$$

which represents symbolically the following equation

$$\begin{aligned} B(Q', Q; \lambda, \mu) &= B_{II}(Q', Q_I; \mu) B_I(Q_I, Q; \lambda) \\ &= B_I(Q', Q_{II}; \lambda) B_{II}(Q_{II}, Q; \mu) \end{aligned} \quad (2.19)$$

where $B(Q', Q; \lambda, \mu)$ is the Bäcklund gauge in (2.8) with

$$\alpha = \mathbb{1}, \quad \beta = \begin{pmatrix} \lambda(v, t) & 0 \\ 0 & \mu(u, t) \end{pmatrix}. \quad (2.20)$$

2.2 Localized soliton solutions

The most natural choice for the parameters entering (2.16) and (2.17) is of course to take λ and μ constants, say

$$\lambda(v, t) = i\lambda, \quad \mu(u, t) = i\mu, \quad (2.21)$$

where λ and μ are complex constants. Then we choose the starting solution Q of (1.5) to be zero together with its related auxiliary field A (note, however, that the vanishing of Q does not imply the vanishing of A).

With these choices the commutation relation (2.19) becomes

$$B(Q, 0; \lambda, \mu) = B_{II}(Q, Q_I; \mu) B_I(Q_I, 0; \lambda) = B_I(Q, Q_{II}; \lambda) B_{II}(Q_{II}, 0; \mu). \quad (2.22)$$

By equating to zero the coefficients of the powers of ∂_y it results that

$$Q_I = \begin{pmatrix} 0 & 0 \\ r_I & 0 \end{pmatrix}, \quad Q_{II} = \begin{pmatrix} 0 & q_{II} \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad (2.23)$$

where

$$r_I = \rho(u, t) \exp(-i\lambda v), \quad q_{II} = \eta(v, t) \exp(i\mu u), \quad (2.24)$$

$$q = \frac{-q_{II,v} + i\lambda q_{II}}{1 + \frac{1}{4}r_I q_{II}}, \quad r = \frac{r_{I,u} + i\mu r_I}{1 + \frac{1}{4}r_I q_{II}} \quad (2.25)$$

and that the operator \mathcal{I} satisfies the equation

$$\mathcal{I}(Q^2) = \frac{1}{2}Q(Q_I - Q_{II}). \quad (2.26)$$

Here above ρ and η are arbitrary functions of the space variable, that can be represented by the following transforms

$$\rho(u, t) = \iint dk \wedge d\bar{k} e^{-iku} \tilde{\rho}(k, t), \quad (2.27)$$

$$\eta(v, t) = \iint dk \wedge d\bar{k} e^{ikv} \tilde{\eta}(k, t), \quad (2.28)$$

with the explicit time dependencies

$$\tilde{\rho}(k, t) = \tilde{\rho}(k, 0) \exp[i(k^2 + \lambda^2)t], \quad (2.29)$$

$$\tilde{\eta}(k, t) = \tilde{\eta}(k, 0) \exp[-i(k^2 + \mu^2)t], \quad (2.30)$$

where $\tilde{\rho}(k, 0)$ and $\tilde{\eta}(k, 0)$ are arbitrary.

Everything now lies in the choice of these two arbitrary functions. We prove hereafter that the requirement that the solution Q is localized and obeys the reduction condition (1.10) determines uniquely the structure of the arbitrary functions $\tilde{\rho}$ and $\tilde{\eta}$. To that end we redefine

$$\rho(u, 0) = 2e^{-i\mu u} S(\sigma), \quad (2.31)$$

$$\eta(v, 0) = 2e^{i\lambda v} T(\tau), \quad (2.32)$$

with the new variables

$$\sigma = \frac{-i}{\mu - \bar{\mu}} \exp[i(\mu - \bar{\mu})u], \quad (2.33)$$

$$\tau = \frac{-i\sigma_0}{\lambda - \bar{\lambda}} \exp[-i(\lambda - \bar{\lambda})v]. \quad (2.34)$$

In these variables, the reduction condition (1.10) takes the following form

$$[1 + \bar{S}(\sigma)\bar{T}(\tau)] \frac{dS}{d\sigma} = [1 + S(\sigma)T(\tau)] \frac{d\bar{T}}{d\tau}, \quad (2.35)$$

which can be solved explicitly and admits 4 independent solutions. The requirement of the localization selects the only solution

$$\begin{aligned} S &= a\sigma + b, & b\bar{a} - \bar{b}a &= 0, \\ T &= \bar{a}\tau + c, & ca - \bar{c}\bar{a} &= 0, \end{aligned} \quad (2.36)$$

where a, b and c are complex constants. Then, by means of (2.31) and (2.32), we arrive at

$$\tilde{\rho}(k, 0) = \rho[\delta(k - \mu) + \delta(k - \bar{\mu})], \quad (2.37)$$

$$\tilde{\eta}(k, 0) = \eta[\delta(k - \lambda) + \delta(k - \bar{\lambda})]. \quad (2.38)$$

Here above η and ρ are arbitrary constants. The requirement of the reduction condition gives the following constraint on these constants

$$\rho(\mu - \bar{\mu}) = \sigma_0 \bar{\eta}(\lambda - \bar{\lambda}). \quad (2.39)$$

Therefore the localized one-soliton solution to the system (1.5)–(1.8) can be written

$$q = -\frac{2\lambda_{\Im}\eta \exp[i\varphi]}{D}, \quad r = -\frac{2\mu_{\Im}\rho \exp[-i\varphi]}{D}, \quad (2.40)$$

with the following definitions

$$\varphi = \mu_{\Re}u + \lambda_{\Re}v + (\lambda_{\Im}^2 - \lambda_{\Re}^2 + \mu_{\Im}^2 - \mu_{\Re}^2)t, \quad (2.41)$$

$$D = 2\gamma(\cosh \xi_1 + \cosh \xi_2) + \exp(\xi_2), \quad \gamma = \frac{1}{4}\eta\rho, \quad (2.42)$$

$$\xi_1 = -\mu_{\Im}u - \lambda_{\Im}v + 2(\lambda_{\Im}\lambda_{\Re} + \mu_{\Im}\mu_{\Re})t, \quad (2.43)$$

$$\xi_2 = \mu_{\Im}u - \lambda_{\Im}v + 2(\lambda_{\Im}\lambda_{\Re} - \mu_{\Im}\mu_{\Re})t, \quad (2.44)$$

and where λ_{\Re} denotes the real part of λ and λ_{\Im} its imaginary part.

For $\lambda_{\Im}\mu_{\Im} \neq 0$ and $\gamma(1 + \gamma) > 0$, the above formulae describe a two-dimensional bell-shaped envelope of the plane wave $\exp(\pm i\varphi)$, exponentially decreasing in all directions in the (u, v) -plane and moving without deformation at the constant velocity

$$\vec{V} = \begin{pmatrix} 2\mu_{\Re} \\ 2\lambda_{\Re} \end{pmatrix}. \quad (2.45)$$

The initial position of the soliton is arbitrary, in other words we may translate the space variables according to the transformation

$$u \rightarrow u - u_0, \quad v \rightarrow v - v_0, \quad u_0, v_0 \in \mathbb{R}, \quad (2.46)$$

and, consequently, the soliton is defined by means of eight real parameters.

By inserting the value of $\mathcal{I}(Q^2)$ obtained in (2.26) into (2.14), rewritten for the gauge $B(Q, O; \lambda, \mu)$ derived in (2.22), we get the auxiliary field A

$$A = \frac{1}{2}(\partial_x - \sigma_3 \partial_y) \partial_y \log Q^2. \quad (2.47)$$

Finally, from (2.1) by using the gauge operator $B(Q, O; \lambda, \mu)$ we obtain the following eigenfunction relative to the one soliton solution

$$\psi(x, y, k) e^{-ik(\sigma_3 x - y)} = \mathbb{1} - \frac{i}{4} \begin{pmatrix} \frac{r_I q}{k - \lambda} & \frac{2q}{k - \mu} \\ -\frac{2r}{k - \lambda} & \frac{q_{II} r}{k - \mu} \end{pmatrix}. \quad (2.48)$$

The analytic properties of this ψ suggested the special choice for the Spectral Transform introduced in section 3.

2.3 Wave soliton solutions of the Hamiltonian DSI

To obtain soliton solutions of the Hamiltonian DSI equation (1.4) we use the elementary Bäcklund gauge in which we keep the general space-dependence of the parameters as indicated in (2.16) and (2.17). We start from the solution $Q = 0$ as before but, now, we choose an auxiliary field A of the form $\text{diag}(A_{00}^{(1)}(v, t), A_{00}^{(2)}(u, t))$ where $A_{00}^{(1)}$ and $A_{00}^{(2)}$ are two arbitrary functions.

The construction of the soliton solution proceeds in the same way as previously, i.e. by imposing the commutation relation (2.22). By means of the elementary Bäcklund gauge of the first kind $B_I(Q_I, 0; \lambda)$ defined in (2.16) we obtain the following solution of (1.5) (which does not satisfy the reduction)

$$Q_I = \begin{pmatrix} 0 & 0 \\ r_I & 0 \end{pmatrix}, \quad A_I = \begin{pmatrix} A_I^{(1)}(v, t) & 0 \\ 0 & A_I^{(2)}(u, t) \end{pmatrix} \quad (2.49)$$

where

$$r_I(u, v, t) = P(u, t)/D(v, t), \quad (2.50)$$

$$A_I^{(1)}(v, t) = A_{00}^{(1)}(v, t) + 2\partial_v^2 \log D(v, t), \quad (2.51)$$

$$A_I^{(2)}(u, t) = A_{00}^{(2)}(u, t). \quad (2.52)$$

The functions $D(v, t)$ and $P(u, t)$ are solutions of the time dependent Schrödinger equations

$$iD_t + D_{vv} + A_{00}^{(1)} D = 0, \quad (2.53)$$

$$iP_t - P_{uu} + A_{00}^{(2)} P = 0 \quad (2.54)$$

and

$$\lambda(v, t) = \partial_v \log D(v, t). \quad (2.55)$$

By applying now to the above solution the elementary Bäcklund gauge of the second kind $B_{II}(Q, Q_I; \mu)$ we obtain the solution

$$Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad A = \begin{pmatrix} A^{(1)} & 0 \\ 0 & A^{(2)} \end{pmatrix} \quad (2.56)$$

where

$$q = \frac{HD_v - H_v D}{ED + PH/4}, \quad r = -\frac{PE_u - P_u E}{ED + PH/4}, \quad (2.57)$$

$$A^{(1)} = A_{00}^{(1)} + 2\partial_v^2 \log(ED + PH/4), \quad (2.58)$$

$$A^{(2)} = A_{00}^{(2)} - 2\partial_u^2 \log(ED + PH/4). \quad (2.59)$$

The functions $H(v, t)$ and $E(u, t)$ satisfy the time dependent Schrödinger equations

$$iH_t + H_{vv} + A_{00}^{(1)}H = 0, \quad (2.60)$$

$$iE_t - E_{uu} + A_{00}^{(2)}E = 0. \quad (2.61)$$

Note that the function μ and the operator $\mathcal{I} = \frac{1}{2} \text{diag}(\partial_u^{-1}, \partial_v^{-1})$ are not uniquely determined but are related by the equation

$$\mu(u, t) = -\frac{1}{4}\partial_v^{-1}(rq) - \partial_u \log(ED + PH/4). \quad (2.62)$$

The boundary values $A_0^{(1)}$ and $A_0^{(2)}$ defined in (1.4) are computed explicitly from the solutions of the four Schrödinger equations and from their potentials (the inputs $A_{00}^{(1)}$ and $A_{00}^{(2)}$)

$$A_0^{(1)} = A_{00}^{(1)} + \partial_v^2 \log(ED + PH/4)|_{u=-\infty} + \partial_v^2 \log(ED + PH/4)|_{u=+\infty}, \quad (2.63)$$

$$A_0^{(2)} = A_{00}^{(2)} - \partial_u^2 \log(ED + PH/4)|_{v=-\infty} - \partial_u^2 \log(ED + PH/4)|_{v=+\infty}. \quad (2.64)$$

One could get the same solution Q, A by applying the Bäcklund gauge of first and second kind in the reversed order $B_I(Q, Q_{II}; \lambda)B_{II}(Q_{II}, 0; \mu)$. One gets

$$Q_{II} = \begin{pmatrix} 0 & q_{II} \\ 0 & 0 \end{pmatrix}, \quad A_{II} = \begin{pmatrix} A_{II}^{(1)} & 0 \\ 0 & A_{II}^{(2)} \end{pmatrix} \quad (2.65)$$

where

$$q_{II}(u, v, t) = H(v, t)/E(u, t), \quad (2.66)$$

$$A_{II}^{(1)}(v, t) = A_{00}^{(1)}(v, t), \quad (2.67)$$

$$A_{II}^{(2)}(u, t) = A_{00}^{(2)}(u, t) - 2\partial_u^2 \log E(u, t) \quad (2.68)$$

and

$$\mu(u, t) = -\partial_u \log E, \quad (2.69)$$

while λ is not uniquely determined. If one requires that the commutation condition (2.22) is satisfied one gets the same solution Q, A , both λ and μ are determined according to the equations (2.69), (2.55) and the operator \mathcal{I} satisfies the equations

$$\partial_u^{-1}qr = qr_I, \quad \partial_v^{-1}qr = -q_{II}r. \quad (2.70)$$

Let us consider the special case of arbitrary boundary values $A_{00}^{(i)}$ real and moving with constant speed

$$\begin{aligned} A_{00}^{(1)}(v, t) &= A_{00}^{(1)}(v + 2\phi t) \\ A_{00}^{(2)}(u, t) &= A_{00}^{(2)}(u + 2\theta t). \end{aligned} \quad (2.71)$$

If the phases of the complex functions D, H and E, P are chosen to be linear functions it results that

$$\begin{aligned} D &= \mathcal{D} \exp[-i\delta], & H &= \eta \exp[-i\delta], \\ E &= \mathcal{E} \exp[i\epsilon], & P &= \rho \exp[i\epsilon] \end{aligned} \quad (2.72)$$

where

$$\begin{aligned}\mathcal{D} &= \mathcal{D}(v + 2\phi t), & \eta &= \eta(v + 2\phi t), \\ \mathcal{E} &= \mathcal{E}(u + 2\theta t), & \rho &= \rho(u + 2\theta t)\end{aligned}\quad (2.73)$$

and

$$\begin{aligned}\delta &= \phi v + (\phi^2 - \phi_0)t + \delta_0, \\ \epsilon &= \theta u + (\theta^2 - \theta_0)t + \epsilon_0\end{aligned}\quad (2.74)$$

with $\phi_0, \theta_0, \delta_0, \epsilon_0$ real constants. The real functions \mathcal{D}, η and \mathcal{E}, ρ satisfy the Schrödinger equations

$$\begin{aligned}\mathcal{D}_{vv} + (A_{00}^{(1)} - \phi_0)\mathcal{D} &= 0, \\ \eta_{vv} + (A_{00}^{(1)} - \phi_0)\eta &= 0\end{aligned}\quad (2.75)$$

and

$$\begin{aligned}\mathcal{E}_{uu} - (A_{00}^{(2)} + \theta_0)\mathcal{E} &= 0, \\ \rho_{uu} - (A_{00}^{(2)} + \theta_0)\rho &= 0.\end{aligned}\quad (2.76)$$

The solution Q is given by the following formulae

$$q = \frac{W(\eta, \mathcal{D})}{\mathcal{E}\mathcal{D} + \rho\eta/4} \exp[-i(\epsilon + \delta)], \quad r = -\frac{W(\rho, \mathcal{E})}{\mathcal{E}\mathcal{D} + \rho\eta/4} \exp[i(\epsilon + \delta)] \quad (2.77)$$

where W is the wronskian operator. The field A is given by the formulae

$$\begin{aligned}A^{(1)} &= A_{00}^{(1)} + 2\partial_v^2 \log(\mathcal{E}\mathcal{D} + \rho\eta/4), \\ A^{(2)} &= A_{00}^{(2)} - 2\partial_u^2 \log(\mathcal{E}\mathcal{D} + \rho\eta/4)\end{aligned}\quad (2.78)$$

and its boundary values by

$$\begin{aligned}A_0^{(1)} &= A_{00}^{(1)} + \partial_v^2 \log(\mathcal{E}\mathcal{D} + \rho\eta/4)|_{u=-\infty} + \partial_v^2 \log(\mathcal{E}\mathcal{D} + \rho\eta/4)|_{u=+\infty}, \\ A_0^{(2)} &= A_{00}^{(2)} - \partial_u^2 \log(\mathcal{E}\mathcal{D} + \rho\eta/4)|_{v=-\infty} - \partial_u^2 \log(\mathcal{E}\mathcal{D} + \rho\eta/4)|_{v=+\infty}.\end{aligned}\quad (2.79)$$

The reduced case $r = \sigma_0 \bar{q}$ is simply obtained by requiring that, for $a \in \mathbb{R}$,

$$W(\eta, \mathcal{D}) = 2a \quad (2.80)$$

$$W(\rho, \mathcal{E}) = -2\sigma_0 a. \quad (2.81)$$

In conclusion we obtain, in the reduced case, a solution of the DSI equation depending on two arbitrary real functions $A_{00}^{(1)}$ and $A_{00}^{(2)}$.

The one soliton solution exponentially decaying in the plane can be obtained from this general solution by choosing

$$\begin{aligned}A_{00}^{(1)} &\equiv A_{00}^{(2)} \equiv 0, & \lambda_0^2 &= \phi_0, & \mu_0^2 &= \theta_0, \\ \mathcal{D} &= \exp[\lambda_0(v + 2\phi t)], & \eta &= h_0 \cosh[\lambda_0(v + 2\phi t)], \\ \mathcal{E} &= \exp[-\mu_0(u + 2\theta t)], & \rho &= p_0 \cosh[\mu_0(u + 2\theta t)]\end{aligned}\quad (2.82)$$

where $\lambda_0, \mu_0, h_0, p_0$ are real constants satisfying the constraints $\sigma_0 \mu_0 p_0 = \lambda_0 h_0$ and $h_0 p_0 > 0$. However, we are here interested in getting solutions with identically zero boundary values

$$A_0^{(1)} \equiv A_0^{(2)} \equiv 0 \quad (2.83)$$

in the reduced case $r = \sigma_0 \bar{q}$. Because this case has been shown to be integrable in the Hamiltonian sense we call it, for brevity, the Hamiltonian case.

It is convenient to introduce two functions $\eta_0(v + 2\phi t)$ and $\rho_0(u + 2\theta t)$ defined as follows

$$A_{00}^{(1)} = 2\partial_v^2 \log \eta_0, \quad A_{00}^{(2)} = -2\partial_u^2 \log \rho_0. \quad (2.84)$$

Then, to get a solution of the Hamiltonian case we have to solve the complicated non linear system of coupled equations for \mathcal{D} , η , η_0 and \mathcal{E} , ρ , ρ_0

$$\partial_v^2 \log \left[\eta_0 \mathcal{D} \left(1 + \frac{\rho \eta}{4\mathcal{E}\mathcal{D}} \right) \right] \Big|_{u=-\infty} + \partial_v^2 \log \left[\eta_0 \mathcal{D} \left(1 + \frac{\rho \eta}{4\mathcal{E}\mathcal{D}} \right) \right] \Big|_{u=+\infty} = 0, \quad (2.85)$$

$$\partial_u^2 \log \left[\rho_0 \mathcal{E} \left(1 + \frac{\rho \eta}{4\mathcal{E}\mathcal{D}} \right) \right] \Big|_{v=-\infty} + \partial_u^2 \log \left[\rho_0 \mathcal{E} \left(1 + \frac{\rho \eta}{4\mathcal{E}\mathcal{D}} \right) \right] \Big|_{v=+\infty} = 0. \quad (2.86)$$

Therefore we have to solve the non linear system of coupled equations (2.75), (2.76), (2.85) and (2.86) with the constraints (2.80) and (2.81). We add to these equations the additional constraints

$$\lim_{u \rightarrow \pm\infty} \frac{\rho}{\mathcal{E}} = -2(\rho_1 \pm \rho_2), \quad (2.87)$$

$$\lim_{v \rightarrow \pm\infty} \frac{\eta}{\mathcal{D}} = -2(\eta_1 \pm \eta_2) \quad (2.88)$$

where ρ_i and η_i are real constants to be determined. These requirements allow us to decouple equations (2.85), (2.75) and (2.80) from equations (2.86), (2.76) and (2.81). Once having found a special solution \mathcal{D} , η , η_0 of the first group of equations and a solution \mathcal{E} , ρ , ρ_0 of the second group one has to verify that they satisfy, respectively, the requirements (2.87) and (2.88).

Let us first consider the equations (2.85), (2.75), (2.80) and (2.87). From (2.85) and (2.87) we get, by using the indeterminacy in the definition of η_0 in (2.84),

$$\eta_0^2 \left[\left(\mathcal{D} - \frac{1}{2}\rho_1\eta \right)^2 - \left(\frac{1}{2}\rho_2\eta \right)^2 \right] = 1. \quad (2.89)$$

If we express \mathcal{D} in terms of a new function $\alpha(v + \phi t)$ as follows

$$\mathcal{D} = \frac{1}{2}\rho_1\eta + \sigma' \frac{1}{2}\rho_2\eta \coth \alpha, \quad \sigma'^2 = 1, \quad (2.90)$$

we obtain

$$\eta = \frac{2 \sinh \alpha}{\rho_2 \eta_0}, \quad (2.91)$$

$$\mathcal{D} = \frac{\rho_1 \sinh \alpha + \sigma' \rho_2 \cosh \alpha}{\rho_2 \eta_0} \quad (2.92)$$

where α and η_0 are to be determined by requiring that (2.75) and (2.80) are satisfied. It results that η_0 decouples from α

$$\partial_v^2 \eta_0 + a^2 \rho_2^2 \eta_0^5 - \phi_0 \eta_0 = 0 \quad (2.93)$$

and α can be determined in terms of η_0

$$\partial_v \alpha = \sigma' a \rho_2 \eta_0^2. \quad (2.94)$$

For solving the equations (2.86), (2.76) and (2.81) we introduce, in an analogous way, a new function $\beta(u + 2\theta t)$ as follows

$$\mathcal{E} = \frac{1}{2} \eta_1 \rho + \sigma'' \frac{1}{2} \eta_2 \rho \coth \beta, \quad \sigma''^2 = 1. \quad (2.95)$$

We get

$$\rho = \frac{2 \sinh \beta}{\eta_2 \rho_0}, \quad (2.96)$$

$$\mathcal{E} = \frac{\eta_1 \sinh \beta + \sigma'' \eta_2 \cosh \beta}{\eta_2 \rho_0} \quad (2.97)$$

where ρ_0 and β are determined by the equations

$$\partial_u^2 \rho_0 + a^2 \eta_2^2 \rho_0^5 - \theta_0 \rho_0 = 0, \quad (2.98)$$

$$\partial_u \beta = -\sigma_0 \sigma'' a \eta_2 \rho_0^2. \quad (2.99)$$

Finally one has to verify that the following consistency conditions are satisfied

$$\lim_{v \rightarrow \pm \infty} (\rho_1 + \sigma' \rho_2 \coth \alpha) = -(\eta_1 \pm \eta_2)^{-1}, \quad (2.100)$$

$$\lim_{u \rightarrow \pm \infty} (\eta_1 + \sigma'' \eta_2 \coth \beta) = -(\rho_1 \pm \rho_2)^{-1}. \quad (2.101)$$

The solution q can be written as

$$q = \frac{2a\rho_2\eta_2\rho_0\eta_0 \exp[-i(\epsilon + \delta)]}{\sinh \alpha \sinh \beta + (\rho_1 \sinh \alpha + \sigma' \rho_2 \cosh \alpha)(\eta_1 \sinh \beta + \sigma'' \eta_2 \cosh \beta)}. \quad (2.102)$$

The ordinary differential equations (2.93) and (2.98) for η_0 and ρ_0 can be explicitly integrated in terms of elementary or classical transcendental functions and, consequently, it is easy to verify the consistency conditions (2.100) and (2.101).

For the sake of definiteness we consider two cases

$$\begin{aligned} \text{(i)} \quad & \partial_v \eta_0 \equiv 0, & \partial_u \rho_0 \equiv 0, & \phi_0 = \lambda_0^2 > 0, & \theta_0 = \mu_0^2 > 0; \\ \text{(ii)} \quad & \partial_v \eta_0 \not\equiv 0, & \partial_u \rho_0 \not\equiv 0, & \phi_0 < 0, & \theta_0 < 0. \end{aligned} \quad (2.103)$$

In the case (i)

$$\eta_0^2 = \frac{\sigma' \lambda_0}{a \rho_2}, \quad \rho_0^2 = -\frac{\sigma_0 \sigma'' \mu_0}{a \eta_2} \quad (2.104)$$

and the consistency conditions are satisfied for

$$\sigma' \sigma'' \lambda_0 \mu_0 > 0, \quad \rho_1 \pm \sigma' \operatorname{sgn}(\lambda_0) \rho_2 = -\frac{1}{\eta_1 \pm \eta_2}. \quad (2.105)$$

If we choose for instance $\sigma'' \lambda_0 > 0$ we get the infinite wave

$$q(u, v, t) = 2|\lambda_0 \mu_0|^{1/2} \frac{\exp[-i(\epsilon + \delta)]}{\cosh \xi} \quad (2.106)$$

where

$$\begin{aligned}\xi &= \mu_0(u + 2\theta t - u_0) - \lambda_0(v + 2\phi t - v_0), \\ \epsilon &= \theta u + (\theta^2 - \mu_0^2)t + \epsilon_0, \\ \delta &= \phi v + (\phi^2 - \lambda_0^2)t + \delta_0.\end{aligned}\tag{2.107}$$

In case (ii) equation (2.93) can be integrated once to the equation

$$(\partial_v \eta_0)^2 + \frac{1}{3}a^2 \rho_2^2 \eta_0^6 - \phi_0 \eta_0^2 - \phi_{00} = 0\tag{2.108}$$

with ϕ_{00} an arbitrary constant that we choose less than zero. Its general solution $\eta_0(v + 2\phi t)$ can be expressed in terms of the Weierstrass elliptic function $\wp(v + 2\phi t; g_2, g_3)$ with invariants

$$g_2 = \frac{4}{3}\phi_0^2, \quad g_3 = \frac{4}{3}a^2 \rho_2^2 \phi_{00}^2 - \frac{8}{27}\phi_0^3\tag{2.109}$$

and negative discriminant

$$\Delta(g_2, g_3) = g_2^3 - 27g_3^2 = -48a^2 \rho_2^2 \phi_{00} \left(a^2 \rho_2^2 \phi_{00}^2 - \frac{4}{9}\phi_0^3 \right)\tag{2.110}$$

according to the formula

$$\eta_0^2(v + 2\phi t) = \frac{\phi_{00}}{\wp(v + 2\phi t; g_2, g_3) - \phi_0/3}.\tag{2.111}$$

Note that $\eta_0^2(v)$ for real v is always regular and that there exists a pure imaginary v_0 such that

$$\wp(v_0; g_2, g_3) = \frac{\phi_0}{3}, \quad \partial_v \wp(v_0; g_2, g_3) = i \frac{2}{\sqrt{3}} a \rho_2 \phi_{00}.\tag{2.112}$$

The function $\alpha(v + 2\phi t)$ obtained by integrating (2.94) (the invariants g_2 and g_3 are omitted and α_0 is a constant; ζ and σ are the ζ - and σ -Weierstrass functions)

$$\alpha(v + 2\phi t) = \frac{\sigma' a \rho_2 \phi_{00}}{\partial_v \wp(v_0)} \left[2(v + \phi t) \zeta(v_0) + \log \frac{\sigma(v + 2\phi t - v_0)}{\sigma(v + 2\phi t + v_0)} \right] + \alpha_0\tag{2.113}$$

results to be real and to behave for large u as follows

$$\alpha(v + 2\phi t) \longrightarrow 2\sigma' a \rho_2 \phi_{00} \frac{\zeta(v_0)}{\partial_v \wp(v_0)} v.\tag{2.114}$$

Analogously we get for $\rho_0^2(u + 2\theta t)$ ($\theta_0 < 0$)

$$\rho_0^2(u + 2\theta t) = \frac{\theta_{00}}{\wp(u + 2\theta t; h_2, h_3) - \theta_0/3}\tag{2.115}$$

where the Weierstrass function $\wp(u; h_2, h_3)$ has invariants

$$h_2 = \frac{4}{3}\theta_0^2, \quad h_3 = \frac{4}{3}a^2 \eta_2^2 \theta_{00}^2 - \frac{8}{27}\theta_0^3,\tag{2.116}$$

negative discriminant

$$\Delta(h_2, h_3) = h_2^3 - 27h_3^2 = -48a^2\eta_2^2\theta_{00} \left(a^2\eta_2^2\theta_{00}^2 - \frac{4}{9}\theta_0^3 \right) \quad (2.117)$$

and

$$\wp(u_0; h_2, h_3) = \frac{\theta_0}{3}, \quad \partial_u \wp(u_0; h_2, h_3) = i \frac{2}{\sqrt{3}} a \eta_2 \theta_{00} \quad (2.118)$$

with u_0 pure imaginary.

The function $\beta(u + 2\theta t)$ is given by

$$\beta(u + 2\theta t) = \frac{\sigma'' \sigma_0 a \eta_2 \theta_{00}}{\partial_u \wp(u_0)} \left[2(u + \theta t) \zeta(u_0) + \log \frac{\sigma(u + 2\theta t - u_0)}{\sigma(u + 2\theta t + u_0)} \right] + \beta_0 \quad (2.119)$$

(invariants h_2 and h_3 are omitted), which is real and has the following behaviour at large u

$$\beta(u + 2\theta t) \longrightarrow -2\sigma_0 \sigma'' a \eta_2 \theta_{00} \frac{\zeta(u_0)}{\partial_u \wp(u_0)} u. \quad (2.120)$$

The consistency conditions (2.100) and (2.101) are satisfied for

$$\sigma' \sigma'' \rho_2 \theta_{00} \phi_{00} \frac{\zeta(v_0) \zeta(u_0)}{\partial_v \wp(v_0) \partial_u \wp(u_0)} > 0 \quad (2.121)$$

and

$$\rho_1 \pm \operatorname{sgn} \left(a \rho_2 \phi_{00} \frac{\zeta(v_0)}{\partial_v \wp(v_0)} \right) \rho_2 = \frac{1}{\eta_1 \pm \eta_2}. \quad (2.122)$$

By inserting these values and functions in the equation (2.102) for q we get an infinite wave with a periodically modulated amplitude.

3 Solitons via the Spectral Transform

3.1 The Spectral Transform of the Kadomtsev–Petviashvili I equation

It turns out that the Spectral Transform for the Kadomtsev–Petviashvili I equation plays a relevant role in the study of the DSI equation. Therefore, this section is dedicated to the main properties of this Spectral Transform that are of interest in this respect. Specifically, we will derive the multi-wave-soliton solution of the Kadomtsev–Petviashvili I equation and the orthogonality relations for the eigenfunctions of the associated spectral problem.

We consider the Kadomtsev–Petviashvili equation in its variant (called KPI)

$$(u_t - 6uu_x + u_{xxx})_x = 3u_{yy} \quad (3.1)$$

with $u = u(x, y, t)$ real. Its Spectral Transform is defined via the associated “time” (the space variable y plays here the role of time) dependent Schrödinger equation

$$-i\Phi_y + \Phi_{xx} - u\Phi = 0. \quad (3.2)$$

The spectral parameter k is introduced by requiring that

$$\Phi(x, y, k) e^{ikx - ik^2 y} = 1 + O\left(\frac{1}{k}\right), \quad k \rightarrow \infty. \quad (3.3)$$

We consider, first, the case in which u is going sufficiently fast to zero at large distances in the (x, y) plane. Then, the eigenfunction Φ can be chosen to be bounded in the (x, y) plane and sectionally holomorphic in the complex k -plane. More precisely, Φ that is called the Jost solution is analytic in the upper and in the lower half plane and its boundary values Φ^\pm on the two sides $\pm \text{Im } k > 0$ of the real k -axis are given by the integral equation

$$\Phi^\sigma(x, y, k) = \int dp e^{-ipx + ip^2 y} R^\sigma(y, k, p), \quad \sigma = \pm, \quad (3.4)$$

where

$$\begin{aligned} R^\sigma(y, k, p) &= \delta(k - p) - \frac{\sigma}{2\pi i} \text{sgn}(p - k) \\ &\times \int d\eta \theta(\sigma(y - \eta)(p - k)) \int d\xi e^{ip\xi - ip^2 \eta} u(\xi, \eta) \Phi^\sigma(\xi, \eta, k). \end{aligned} \quad (3.5)$$

When it is not differently indicated the integration is performed all along the real axis from $-\infty$ to $+\infty$.

The Spectral Transform $F(k, l)$ of the potential $u(x, y)$ is defined as the measure of the departure from analyticity of the Jost solution Φ

$$\frac{\partial \Phi}{\partial \bar{k}} = \int \int dl \wedge d\bar{l} \Phi(l) F(k, l) \quad (3.6)$$

where

$$\frac{\partial}{\partial \bar{k}} \equiv \frac{1}{2} \left(\frac{\partial}{\partial k_{\Re}} + i \frac{\partial}{\partial k_{\Im}} \right), \quad k_{\Re} = \text{Re } k, \quad k_{\Im} = \text{Im } k \quad (3.7)$$

is the so called $\bar{\partial}$ -derivative.

The main quantity to study in order to get complete information on the Spectral Transform of the KPI equation is the scalar product

$$\langle \Phi^\sigma(k), \Phi^{\sigma'}(p) \rangle \equiv \frac{1}{2\pi} \int dx \Phi^\sigma(x, y, k) \overline{\Phi^{\sigma'}(x, y, p)}. \quad (3.8)$$

By inserting here the integral equation for Φ^σ and $\Phi^{\sigma'}$ we get

$$\langle \Phi^\sigma(k), \Phi^{\sigma'}(p) \rangle = \left(\mathbf{R}^\sigma(y) \mathbf{R}^{\sigma'\dagger}(y) \right)(k, p), \quad (3.9)$$

where $R^\sigma(y, k, p)$ is considered as the kernel of an integral operator $\mathbf{R}^\sigma(y)$, $R^{\sigma\dagger}(y, k, p) = \overline{R^\sigma(y, p, k)}$ is the kernel of the adjoint operator $\mathbf{R}^{\sigma\dagger}(y)$ and

$$\left(\mathbf{R}^\sigma(y) \mathbf{R}^{\sigma'\dagger}(y) \right)(k, p) = \int dl R^\sigma(y, k, l) \overline{R^{\sigma'}(y, p, l)} \quad (3.10)$$

is the kernel of the product $\mathbf{R}^\sigma(y)\mathbf{R}^{\sigma'\dagger}(y)$. By differentiating the scalar product (3.8) with respect to y and by using (3.2) for Φ^σ and $\Phi^{\sigma'}$ one proves that it is y independent. For exploiting this information it is convenient to introduce

$$R_\pm^\sigma(k, p) = \lim_{y \rightarrow \pm\infty} R^\sigma(y, k, p). \quad (3.11)$$

They are kernels of the triangular integral operators \mathbf{R}_\pm^σ whose explicit expressions

$$R_\pm^\sigma(k, p) = \delta(k - p) \mp \vartheta(\mp\sigma(k - p))r^\sigma(k, p), \quad (3.12)$$

$$r^\sigma(k, p) = \frac{1}{2\pi i} \int d\eta \int d\xi e^{ip\xi - ip^2\eta} u(\xi, \eta) \Phi^\sigma(\xi, \eta, k), \quad \sigma = \pm, \quad (3.13)$$

are obtained by inserting into (3.5) the identity

$$\sigma \operatorname{sgn}(p - k) \vartheta(\sigma(y - \eta)(p - k)) = \mp \vartheta(\mp(y - \eta)) \pm \vartheta(\mp\sigma(k - p)). \quad (3.14)$$

Then, the y independence of the scalar product implies that

$$\langle \Phi^\sigma(k), \Phi^{\sigma'}(p) \rangle = (\mathbf{R}_+^\sigma \mathbf{R}_+^{\sigma'\dagger})(k, p) = (\mathbf{R}_-^\sigma \mathbf{R}_-^{\sigma'\dagger})(k, p). \quad (3.15)$$

In the case $\sigma' = -\sigma$ the two operators $\mathbf{R}_+^\sigma \mathbf{R}_+^{-\sigma\dagger}$ and $\mathbf{R}_-^\sigma \mathbf{R}_-^{-\sigma\dagger}$ are, respectively, lower and upper triangular or upper and lower triangular according to the sign, positive or negative, of σ . Consequently, the equality in (3.15) implies that

$$\mathbf{R}_+^\sigma \mathbf{R}_+^{-\sigma\dagger} = \mathbf{R}_-^\sigma \mathbf{R}_-^{-\sigma\dagger} = \mathbf{I}, \quad (3.16)$$

where the kernel of the unity operator is $I(k, p) = \delta(k - p)$, so that Φ^σ and $\Phi^{-\sigma}$ are orthogonal

$$\langle \Phi^\sigma(k), \Phi^{-\sigma}(p) \rangle = \delta(k - p). \quad (3.17)$$

In view of the relevant role played by \mathbf{R}_\pm^σ it is convenient, by using again the identity (3.14) and the definition (3.11), to recast the integral equation (3.4) in the form

$$\begin{aligned} \Phi^\sigma(x, y, k) &= \int dp e^{-ipx + ip^2y} R_\pm^\sigma(k, p) \\ &\quad - \frac{1}{2\pi i} \int dp \int_{\pm\infty}^y d\eta \int d\xi e^{ip(\xi - x) - ip^2(\eta - y)} u(\xi, \eta) \Phi^\sigma(\xi, \eta, k). \end{aligned} \quad (3.18)$$

Let us now turn to the eigenfunctions Ψ_\pm solution of the integral equation

$$\Psi_\pm(x, y, k) = e^{-ikx + ik^2y} - \frac{1}{2\pi i} \int dp \int_{\pm\infty}^y d\eta \int d\xi e^{ip(\xi - x) - ip^2(\eta - y)} u(\xi, \eta) \Psi_\pm(\xi, \eta, k). \quad (3.19)$$

From (3.18) we see that

$$\Phi^\sigma = \mathbf{R}_\pm^\sigma \Psi_\pm. \quad (3.20)$$

As we know by (3.16) the operators $\mathbf{R}_\pm^{-\sigma\dagger}$ are right inverse of the \mathbf{R}_\pm^σ . Let the potential u be such that these operators are both sides mutually inverse, i.e. let be besides (3.16)

$$\mathbf{R}_+^{-\sigma\dagger} \mathbf{R}_+^\sigma = \mathbf{R}_-^{-\sigma\dagger} \mathbf{R}_-^\sigma = \mathbf{I}. \quad (3.21)$$

Then the relation (3.20) can be rewritten as

$$\Psi_{\pm} = \mathbf{R}_{\pm}^{-\sigma \dagger} \Phi^{\sigma}. \quad (3.22)$$

Due to the defining integral equation (3.19) the solutions Ψ_{\pm} are σ independent. Consequently equation (3.22) yields

$$\mathbf{R}_{\pm}^{-\sigma \dagger} \Phi^{\sigma} = \mathbf{R}_{\pm}^{\sigma \dagger} \Phi^{-\sigma} \quad (3.23)$$

and, by using again (3.16),

$$\Phi^{\sigma} = \mathcal{F}^{-\sigma} \Phi^{-\sigma}, \quad (3.24)$$

where

$$\mathcal{F}^{-\sigma} = \mathbf{R}_{\pm}^{\sigma} \mathbf{R}_{\pm}^{\dagger}. \quad (3.25)$$

From (3.15) and (3.25) we have directly that

$$\langle \Phi^{\sigma}(k), \Phi^{\sigma}(p) \rangle = \mathcal{F}^{-\sigma}(k, p). \quad (3.26)$$

It is convenient to separate the δ distribution contained in the kernel of \mathcal{F}^{σ} by writing

$$\mathcal{F}^{\sigma}(k, p) = \delta(k - p) - \sigma f^{\sigma}(k, p). \quad (3.27)$$

Then, equations (3.24) and (3.25) read

$$\Phi^{+}(x, y, k) - \Phi^{-}(x, y, k) = \int dl f^{\sigma}(k, l) \Phi^{\sigma}(x, y, l), \quad (3.28)$$

$$f^{-\sigma}(k, p) = r^{\sigma}(k, p) \vartheta(k - p) + \overline{r^{\sigma}(p, k)} \vartheta(p - k) + \sigma \int_{-\infty}^k dl \vartheta(p - l) r^{\sigma}(k, l) \overline{r^{\sigma}(p, l)}. \quad (3.29)$$

By recalling that on the real k -axis

$$\frac{\partial \Phi}{\partial k}(k) = \frac{i}{2} (\Phi^{+}(k) - \Phi^{-}(k)) \delta(k_{\Im}) \quad (3.30)$$

we get for the Spectral Transform of u defined in (3.6)

$$F(k, l) = \frac{i}{2} f^{\sigma}(k, l) \delta(k_{\Im}). \quad (3.31)$$

Formula (3.29) solves the direct spectral problem furnishing an explicit expression of the Spectral Transform in terms of u and of the Jost solution Φ^{σ} via $r^{\sigma}(k, p)$.

We note that the operators $\mathcal{F}^{\sigma} (\sigma = \pm)$, which yield the Spectral Transform of u , are selfadjoint

$$\mathcal{F}^{\sigma \dagger} = \mathcal{F}^{\sigma} \quad (3.32)$$

and one is the inverse of the other

$$\mathcal{F}^{\sigma} \mathcal{F}^{-\sigma} = \mathbf{I}. \quad (3.33)$$

These two equations can be considered as the characterization equations for the Spectral Transform \mathcal{F}^{σ} (in the sense that they determine the class of admissible spectral data)

if \mathcal{F}^σ is the product, as in (3.16), of two operators \mathbf{R}_\pm^σ which have the triangular form indicated in (3.12). Because the characterization equations are more simply expressed in terms of \mathcal{F}^σ we prefer, in the following, to define \mathcal{F}^σ as the Spectral Transform of u .

The reconstruction of the potential u can be performed starting from the Spectral Data \mathcal{F}^σ . First, one solves the singular Fredholm equation

$$\Phi^\sigma(x, y, k) = e^{-ikx + ik^2 y} - \frac{\sigma'}{2\pi i} \int dq \frac{e^{-i(k-q)x + i(k^2 - q^2)y}}{q - k - i0\sigma} \int dp [\mathcal{F}^{\sigma'} - \mathbf{I}](q, p) \Phi^{\sigma'}(x, y, p). \quad (3.34)$$

The obtained Φ^σ , by the Cauchy-Green theorem, solves the non local Riemann-Hilbert problem expressed by the $\bar{\partial}$ -equation (3.6) (or its equivalent form (3.24)) and satisfies the asymptotic requirement (3.3). Secondly, one expands in powers of $1/k$ the fraction $1/(q - k - i0\sigma)$ in (3.34) and inserts the obtained asymptotic expansion in (3.2). We have

$$u(x, y) = -\sigma' \frac{1}{\pi} \frac{\partial}{\partial x} \int dq \int dp [\mathcal{F}^{\sigma'} - \mathbf{I}](q, p) \Phi^{\sigma'}(x, y, p) e^{iqx - iq^2 y}. \quad (3.35)$$

In the following section we are interested also in potentials u describing N interacting wave solitons. Then, u goes to a constant along N directions in the (x, y) plane and the Green function in the integral equation (3.4) has to be corrected in order to avoid divergences. The theory of the Spectral Transform does not result to be substantially different from the theory in the case of u vanishing at large distances. However, there are many subtle technical difficulties to handle which are out of the scope of the present paper. Therefore, we restrict the discussion to the main points and we refer for details to the published papers by two of the authors of this paper (M. B. and F. P.) and by A. Pogrebkov and M. Polivanov.

The Jost solution $\nu(x, y, k) = \Phi(x, y, k) \exp[ikx - ik^2 y]$ in the complex k -plane can be defined as the solution of the following integral equation

$$\begin{aligned} \nu(x, y, k) &= 1 + \frac{1}{(2\pi)^2} \iint dp dq \frac{1}{q - (p - i0)(p + 2k)} \\ &\times \int d\eta d\xi \nu(\xi, \eta, k) u(\xi, \eta) e^{ip(\xi - x) - iq(\eta - y)} \end{aligned} \quad (3.36)$$

where the integrations must be done in the order indicated from the right to the left. For a potential u going sufficiently fast to zero for large $x^2 + y^2$ the order is unessential and one can explicitly perform, first, the integration over q recovering the integral equation (3.4).

The integral equation (3.36) contains as a special case the unidimensional case. In fact if $u(x, y) = \tilde{u}(x)$ by introducing

$$\tilde{\Phi}(x, k) = e^{-ik^2 y} \Phi(x, y, k) \quad (3.37)$$

one recovers the stationary Schrödinger equation

$$\tilde{\Phi}_{xx}(x, k) + (k^2 - \tilde{u}(x)) \tilde{\Phi}(x, k) = 0. \quad (3.38)$$

Moreover, in (3.36), once integrated over η , the integration over ξ and p, q can be interchanged and one obtains by computing the limit $k_{\Im} \rightarrow 0$ on the two sides of the real

k-axis the familiar integral equations

$$\begin{aligned}\tilde{\Phi}^+(x, k) &= e^{-ikx} \\ &+ \int d\xi \frac{e^{ik|x-\xi|}}{2i(k+i0)} \tilde{u}(\xi) \tilde{\Phi}^+(\xi, k)\end{aligned}\quad (3.39)$$

$$\begin{aligned}\tilde{\Phi}^-(x, k) &= e^{-ikx} \\ &- \int d\xi \theta(\xi - x) \frac{\sin[k(x - \xi)]}{k - i0} \tilde{u}(\xi) \tilde{\Phi}^-(\xi, k)\end{aligned}\quad (3.40)$$

defining the sectionally meromorphic Jost solution $\tilde{\Phi}$ of the stationary Schrödinger equation.

In the general bidimensional case, as in the previous case (u vanishing at infinity), the main quantities to study are the scalar products of the Jost solutions. They result to be still y independent and, therefore, formula (3.15) remains valid. The most difficult point is the computation of the limits R_{\pm}^{σ} and of the characterization equations. It results that formula (3.12) must be changed as follows

$$R_{\pm}^{\sigma}(k, p) = \delta(k - p)(1 + Z_{\pm}^{\sigma}(k)) \mp \theta(\mp \sigma(k - p))r^{\sigma}(k, p) \quad (3.41)$$

by adding to the coefficient of the δ distribution a function $Z_{\pm}^{\sigma}(k)$ to be determined.

The Jost solutions Φ^{σ} and $\Phi^{-\sigma}$ are still orthogonal and this is the main property we will use in the following section.

Let us now proceed and consider the N -soliton solution of the KPI, namely the potential u characterized by a Spectral Transform containing exclusively a number of discrete eigenvalues $\mu_n \in \mathbb{C}$ ($n = 1, \dots, N$). For real u this spectrum $f_d(k, l)$ must satisfy the characterization equation

$$\overline{f}_d(k, l) = f_d(l, k) \quad (3.42)$$

and therefore its most general form is

$$f_d(k, l) = \sum_{n=1}^N \sum_{m=1}^N r_{nm} \delta(l - \overline{\mu}_m) \delta(k - \mu_n) \quad (3.43)$$

with r_{nm} an arbitrary constant hermitian matrix. It is convenient to parameterize this matrix as follows

$$r_{nm} = 2\pi i \exp[i\overline{\mu}_m(x_{om} - \overline{\mu}_m y_{om}) - i\mu_n(x_{on} - \mu_n y_{on})] C_{nm} \sharp_{mm} \quad (3.44)$$

where the real parameters x_{on} and y_{on} fix the initial position of the n^{th} soliton in the plane,

$$\sharp_{nm} \equiv \overline{\mu}_n - \mu_m, \quad (3.45)$$

and the complex matrix C satisfies

$$C_{nn}^2 = 1, \quad C_{nm} \sharp_{mm} = \overline{C}_{mn} \sharp_{nn}. \quad (3.46)$$

For definiteness we choose $C_{nn} = 1$ and $\text{Im } \mu_n > 0$.

Because the Jost solution $\Phi(x, y, k)$ in the pure discrete case has only simple poles at $k = \mu_n$ ($n = 1, \dots, N$) and satisfies the asymptotic property (3.3) it admits the representation

$$\Phi(x, y, k) = e^{-ikx + ik^2 y} \left(1 + \sum_{n=1}^N \frac{\varphi_n(x, y)}{k - \mu_n} e^{\alpha_n} \right) \quad (3.47)$$

where, for convenience, it has been introduced an exponential factor e^{α_n} with

$$\alpha_n = i\mu_n(x - x_{on}) - i\mu_n^2(y - y_{on}) \quad (3.48)$$

The insertion of this representation into the $\bar{\partial}$ -equation (3.6) yields an algebraic equation for the functions φ_n

$$\sum_{m=1}^N \mathcal{A}_{nm} \varphi_m = - \sum_{m=1}^N C_{nm} \#_{mm} e^{\bar{\alpha}_m} \quad (3.49)$$

where

$$\mathcal{A} = 1 + C\alpha, \quad \alpha_{nm} = \frac{\#_{nn}}{\#_{nm}} e^{\bar{\alpha}_n + \alpha_m}, \quad (3.50)$$

while its insertion into (3.35) yields the potential u

$$u = -2i\partial_x \sum_{n=1}^N \varphi_n e^{\alpha_n}. \quad (3.51)$$

To see that (3.49) can be solved for φ_n it is sufficient to show that \mathcal{A} is positive definite. In the following section in the more general framework of the Spectral Transform for the DSI equation we prove that this is true if the hermitian matrix $iC_{nm}\#_{mm}$ has positive eigenvalues.

By the rule for differentiating a determinant the N soliton solution u can be written in the closed form

$$u = -2\partial_x^2 \ln \det \mathcal{A} \quad (3.52)$$

and the orthogonality relations for the Jost solutions

$$i \int dx \overline{\Phi^-(x, y, \bar{\mu}_m)} \text{Res}(\Phi^+(x, y, k), \mu_n) = \delta_{mn} \quad (3.53)$$

can be derived.

The potential $u(x, y)$ is going to a constant at large distance along the directions $x - 2\text{Re } \mu_n y = \text{const}$ and describes N intersecting waves of infinite length.

3.2 The Spectral Transform of the Davey-Stewartson I equation

We consider the DSI equation (1.5) with boundary conditions defined as in (1.8). The real boundaries $a_0^{(1)}(v, t)$ and $a_0^{(2)}(u, t)$ are assumed to go to zero at large distances in the (v, t) and (u, t) plane, respectively, with the possible exception of a finite number of directions along which they are going to some constants.

According to the usual scheme, in order to linearize the DSI equation, we have to define the Spectral Transform for the Zakharov-Shabat spectral problem in the plane (hyperbolic case)

$$T_1 \psi \equiv (\partial_x + \sigma_3 \partial_y + Q) \psi = 0 \quad (3.54)$$

The complex spectral parameter k is introduced by requiring that the 2×2 matrix Jost solution ψ satisfies the asymptotic property

$$\psi(x, y, k) e^{-ik(\sigma_3 x - y)} = \mathbb{1} + O\left(\frac{1}{k}\right), \quad k \rightarrow \infty \quad (3.55)$$

The Green function of the Zakharov–Shabat spectral problem can be chosen to be sectionally holomorphic and the values ψ^\pm of ψ on the two sides $\pm \text{Im } k > 0$ of the real axis in the k -plane are given by the integral equations

$$\psi^\pm = \psi_0^\pm + G^\pm \psi^\pm \quad (3.56)$$

where

$$\begin{aligned} G^\pm \psi &= (G_1^\pm \psi_1, G_2^\pm \psi_2) \\ \psi &\equiv (\psi_1, \psi_2) \equiv \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} \end{aligned} \quad (3.57)$$

with

$$\begin{aligned} G_1^\pm &= \frac{1}{2} \begin{pmatrix} 0 & \int_u^{-\infty} du' q(u', v) \\ \int_v^{\pm\infty} dv' r(u, v') & 0 \end{pmatrix} \\ G_2^\pm &= \frac{1}{2} \begin{pmatrix} 0 & \int_u^{\mp\infty} du' q(u', v) \\ \int_v^{-\infty} dv' r(u, v') & 0 \end{pmatrix} \end{aligned} \quad (3.58)$$

and $\psi_0^\pm = \text{diag}(\psi_{01}^\pm, \psi_{02}^\pm)$ is an arbitrary solution of the homogeneous part of the Zakharov–Shabat spectral equation

$$(\partial_x + \sigma_3 \partial_y) \psi_0^\pm = 0 \quad (3.59)$$

It is worth noting for future use that the Green operators G^\pm have the symmetric property

$$G_2^+ = G_1^- \quad (3.60)$$

and are k independent.

The 2×2 matrix Spectral Transform $R(k, l)$ of $Q(x, y)$ is defined as the measure of the departure from analyticity of ψ

$$\frac{\partial}{\partial \bar{k}} \psi(x, y, k) = \iint_{\mathbf{C}} dl \wedge d\bar{l} \psi(x, y, l) R(k, l). \quad (3.61)$$

In contrast with the one dimensional case, once chosen the Green operator, the asymptotic requirement (3.55) does not fix ψ_{01} and ψ_{02} which are arbitrary functions of v and u , respectively (see (3.59)). Therefore, for different choices of ψ_0 we get different ψ and consequently via the definition (3.61) different Spectral Transforms $R(k, l)$.

We search for a Spectral Transform that satisfies the following two fundamental properties:

- i) its time evolution can be explicitly integrated;
- ii) the discrete part of the spectrum corresponds to solitons and the continuous part to radiation.

In order to satisfy the requirement $i)$, in analogy with the one dimensional case, we fix the time evolution of ψ by requiring that

$$T_2\psi = \psi\Omega(k) \quad (3.62)$$

where T_2 is the second Lax operator in (1.13) and $\Omega(k) = -\sigma_3 k^2$ is the dispersive function of the DSI equation. In fact, by applying to both sides of (3.62) the operator $\frac{\partial}{\partial k}$ and by using the definition of the Spectral Transform in (3.61) we get the linear time evolution equation for the Spectral Transform

$$iR_t(k, l, t) = R(k, l, t)\Omega(k) - \Omega(l)R(k, l, t) \quad (3.63)$$

which is easily explicitly solved getting

$$R(k, l, t) = e^{i\Omega(l)t} R(k, l, 0) e^{-i\Omega(k)t} \quad (3.64)$$

By taking the limit ($u \rightarrow -\infty$, v fixed) for the $(\)_{11}$ matrix element of (3.62) and the limit ($v \rightarrow -\infty$, u fixed) for the $(\)_{22}$ element we derive two time dependent Schrödinger equations for ψ_{01} and ψ_{02}

$$\begin{aligned} [\partial_v^2 + k^2 + a_0^{(1)}(v, t)]\psi_{01}(v, t, k) &= -i\partial_t\psi_{01}(v, t, k) \\ [\partial_u^2 + k^2 - a_0^{(2)}(u, t)]\psi_{01}(u, t, k) &= i\partial_t\psi_{02}(u, t, k) \end{aligned} \quad (3.65)$$

These equations together with the asymptotic property

$$\psi_0 e^{-ik(\sigma_3 x - y)} = \mathbb{1} + O\left(\frac{1}{k}\right), \quad k \rightarrow \infty, \quad (3.66)$$

which can be derived from (3.56) by using (3.55), uniquely determine ψ_0 in terms of the boundary values $a_0^{(1)}$ and $a_0^{(2)}$. Precisely, we have

$$\psi_{01}(v, t, k) = \Phi^{(1)}(v, t, k) e^{ik^2 t} \quad (3.67)$$

$$\psi_{02}(u, t, k) = \Phi^{(2)}(u, t, k) e^{-ik^2 t} \quad (3.68)$$

where $\Phi^{(1)}(v, t, k)$ and $\Phi^{(2)}(u, t, k)$ are the Jost solutions of the time dependent Schrödinger equations with space variables v and u and potentials $a^{(1)}(v, t)$ and $a^{(2)}(u, t)$, respectively.

The integral equations (3.56) for ψ are of Volterra type and therefore the singularities of ψ in the complex k -plane are those of ψ_0 and of the sectionally holomorphic Green function. Therefore we need to revisit the Spectral Transform for the time dependent Schrödinger equation. In particular if $a_0^{(1)}$ and $a_0^{(2)}$ are wave solitons in the plane (v, t) and (u, t) the eigenfunction ψ_0 and consequently ψ have simple poles in k . The existence and the location of poles are uniquely determined by the boundary values $a_0^{(1)}$ and $a_0^{(2)}$. However, we will see that the boundaries do not characterize completely the discrete spectrum and one is left with an unexpected freedom in choosing other independent parameters.

The continuous part $R_c(k, l)$ of the Spectral Transform $R(k, l)$ measures the discontinuity $\psi_+ - \psi_-$ of ψ along the real k axis and it has the form

$$R_c(k, l) = -\frac{1}{4} \begin{pmatrix} \delta(l_{\Im} + 0) & 0 \\ 0 & \delta(l_{\Im} - 0) \end{pmatrix} \begin{pmatrix} F_1(k, l) & -S_2(k, l) \\ S_1(k, l) & F_2(k, l) \end{pmatrix} \delta(k_{\Im}) \quad (3.69)$$

where we make the choice $\begin{pmatrix} \delta(l_{\mathfrak{S}} + 0) & 0 \\ 0 & \delta(l_{\mathfrak{S}} - 0) \end{pmatrix}$ instead of the usual one $\delta(l_{\mathfrak{S}} + 0)\mathbb{1}$ or $\delta(l_{\mathfrak{S}} - 0)\mathbb{1}$ in order to exploit in the following the symmetry $G_2^+ = G_1^-$ of the Green operator.

Let us introduce the integral operator

$$G = (G_1^-, G_2^+). \quad (3.70)$$

From the integral equation (3.56) we derive directly (k is real, space time variables are understood)

$$(\psi^+ - \psi^-)(k) = \mathcal{F}_c(k) + G(\psi^+ - \psi^-)(k) \quad (3.71)$$

where

$$\mathcal{F}_c(k) = \begin{pmatrix} (\psi_{01}^+ - \psi_{01}^-)(k) & -\frac{1}{2} \int du \, q \psi_{22}^-(k) \\ \frac{1}{2} \int dv \, r \psi_{11}^+(k) & (\psi_{02}^+ - \psi_{02}^-)(k) \end{pmatrix}. \quad (3.72)$$

Here and in the following when it is not differently indicated the integration is performed all along the real axis from $-\infty$ to $+\infty$.

By inserting the same integral equation (3.56) into the r.h.s. of (3.61), by using the symmetry $G_2^+ = G_1^-$ and the k independence of G^\pm and by recalling that on the real k axis

$$\frac{\partial \psi}{\partial \bar{k}}(k) = \frac{i}{2}(\psi^+(k) - \psi^-(k))\delta(k_{\mathfrak{S}}) \quad (3.73)$$

we get

$$(\psi^+ - \psi^-)(k) = \tilde{\mathcal{F}}_c(k) + G(\psi^+ - \psi^-)(k) \quad (3.74)$$

where

$$\tilde{\mathcal{F}}_c(k) = \int dl \begin{pmatrix} \psi_{01}^-(l)F_1(k, l) & -\psi_{01}^-(l)S_2(k, l) \\ \psi_{02}^+(l)S_1(k, l) & \psi_{02}^+(l)F_2(k, l) \end{pmatrix} \quad (3.75)$$

Since the operator G is of Volterra type the related homogeneous integral equation has only the vanishing solution and, consequently, by comparing the two integral equations (3.71) and (3.74) we get

$$\mathcal{F}_c = \tilde{\mathcal{F}}_c. \quad (3.76)$$

From this equation we deduce that F_1 and F_2 are the continuous component of the Spectral Transform of the boundaries $a_0^{(1)}$ and $a_0^{(2)}$ and, by using the orthogonality relations (3.17) rewritten for the eigenfunctions ψ_{01} and ψ_{02} , we express explicitly S_1 and S_2 in terms of Q , ψ and ψ_0 as follows

$$\begin{aligned} S_1(k, l) &= \frac{1}{4\pi} \iint du \, dv \, r(u, v) \psi_{11}^+(u, v, k) \overline{\psi_{02}^-(u, l)} \\ S_2(k, l) &= \frac{1}{4\pi} \iint du \, dv \, q(u, v) \psi_{22}^-(u, v, k) \overline{\psi_{01}^+(v, l)} \end{aligned} \quad (3.77)$$

In order to get the characterization equation satisfied by the spectral data $S_i(k, l)$ in the reduced case $r = \sigma_0 \bar{q}$ we consider the two Jost solutions

$$\begin{aligned} \psi_1^+(l) &= \begin{pmatrix} \psi_{01}^+(l) \\ 0 \end{pmatrix} + G_1^+ \psi_1^+(l) \\ \psi_2^-(k) &= \begin{pmatrix} 0 \\ \psi_{02}^-(k) \end{pmatrix} + G_2^- \psi_2^-(k) \end{aligned} \quad (3.78)$$

and we note that from (3.54), for $r = \sigma_0 \overline{q}$, it easily follows that

$$\left(\psi_{11}^+(l) \overline{\psi_{12}^-(k)} \right)_u = \sigma_0 \left(\psi_{21}^+(l) \overline{\psi_{22}^-(k)} \right)_v. \quad (3.79)$$

By integrating it in the (u, v) plane and by using (3.78) we get

$$\overline{S_2(k, l)} = \sigma_0 S_1(l, k). \quad (3.80)$$

From (3.64) we derive that the spectral data evolve in time as follows

$$S_1(k, l, t) = e^{i(k^2 + l^2)t} S_1(k, l, 0) \quad (3.81)$$

$$S_2(k, l, t) = e^{-i(k^2 + l^2)t} S_2(k, l, 0) \quad (3.82)$$

coherently with the characterization equation (3.80).

The discrete component $R_d(k, l)$ of the Spectral Transform has different possible characterization. We consider two of them whose matrix elements are linear combinations of δ distributions in the complex k -plane and l -plane.

The first one is given by the formula

$$R_d(k, l) = -2\pi i \begin{pmatrix} R_{11}(k, l) & R_{12}(k, l) \\ R_{21}(k, l) & R_{22}(k, l) \end{pmatrix} \quad (3.83)$$

$$\begin{aligned} R_{11}(k, l) &= \sum_{n=1}^N \sum_{n'=1}^N \tau_{11}^{nn'} \delta(l - \overline{\lambda}_{n'}) \delta(k - \lambda_n) \\ R_{12}(k, l) &= \sum_{n=1}^N \sum_{m=1}^M \tau_{12}^{mn} \delta(l - \overline{\lambda}_n) \delta(k - \mu_m) \\ R_{21}(k, l) &= \sum_{n=1}^N \sum_{m=1}^M \tau_{21}^{nm} \delta(l - \overline{\mu}_m) \delta(k - \lambda_n) \\ R_{22}(k, l) &= \sum_{m=1}^M \sum_{m'=1}^M \tau_{22}^{mm'} \delta(l - \overline{\mu}_{m'}) \delta(k - \mu_m) \end{aligned}$$

where λ_n and μ_m , the so called discrete values of the spectrum, are, respectively, the locations of poles in the complex k -plane of the first and second column of the Jost matrix solution ψ and the τ_{ij}^{mn} are some complex constants to be related to the initial value of Q .

The δ distribution in the complex plane is defined as

$$\iint_{\mathbb{C}} dl \wedge d\bar{l} \delta(l - l_0) f(l) = f(l_0) \quad (3.84)$$

If we introduce the distribution $(l - l_0)\delta(l - l_0)$ operating on singular functions in the complex plane as follows

$$\iint_{\mathbb{C}} dl \wedge d\bar{l} (l - l_0) \delta(l - l_0) f(l) = \text{Res}(f, l_0) \quad (3.85)$$

we can deal with a simpler form of the Spectral Transform characterized by an off diagonal matrix

$$\begin{aligned}
R_d(k, l) &= -\pi i \begin{pmatrix} 0 & -R_2(k, l) \\ R_1(k, l) & 0 \end{pmatrix} \\
R_1(k, l) &= \sum_{n=1}^N \sum_{m=1}^M [\tau_1^{nm} (l - \mu_m) \delta(l - \mu_m) \delta(k - \lambda_n) + \tilde{\tau}_1^{nm} \delta(l - \bar{\mu}_m) \delta(k - \lambda_n)] \\
R_2(k, l) &= \sum_{n=1}^N \sum_{m=1}^M [\tau_2^{mn} (l - \lambda_n) \delta(l - \lambda_n) \delta(k - \mu_m) + \tilde{\tau}_2^{mn} \delta(l - \bar{\lambda}_n) \delta(k - \mu_m)].
\end{aligned} \tag{3.86}$$

By inserting the two different characterization of R_d in the $\bar{\partial}$ -equation (3.61) one derives easily that they are equivalent and, more precisely, that the two set of spectral data are related by the following equations

$$\begin{aligned}
\tau_{11} &= -(4 + \tau_1 \tau_2)^{-1} \tau_1 \tilde{\tau}_2 \\
\tau_{12} &= -(4 + \tau_2 \tau_1)^{-1} 2 \tilde{\tau}_2 \\
\tau_{21} &= (4 + \tau_1 \tau_2)^{-1} 2 \tilde{\tau}_1 \\
\tau_{22} &= -(4 + \tau_2 \tau_1)^{-1} \tau_2 \tilde{\tau}_1
\end{aligned} \tag{3.87}$$

and

$$\begin{aligned}
\tau_1 &= 2\tau_{11}\tau_{12}^{-1} \\
\tau_2 &= -2\tau_{22}\tau_{21}^{-1} \\
\tilde{\tau}_1 &= 2\tau_{21} - 2\tau_{11}\tau_{12}^{-1}\tau_{22} \\
\tilde{\tau}_2 &= -2\tau_{12} + 2\tau_{22}\tau_{21}^{-1}\tau_{11}.
\end{aligned} \tag{3.88}$$

Without any loss of generality we can consider $M = N$. The cases $M < N$ and $M > N$ are recovered from the special case $M = N$ by choosing some μ or λ equal.

In order to distinguish between parameters that are fixed by the choice of the boundaries, i.e. the external data given at all times, and parameters that are solely connected to the initial data at time $t = 0$ it is convenient to parameterize the spectral data in (3.86) as follows

$$\begin{aligned}
\tau_1^{nm} &= \rho_{nm} \exp[i\mu_m u_{om} + i\lambda_n v_{on} + i\mu_m^2 t + i\lambda_n^2 t] \\
\tilde{\tau}_1^{nm} &= \sum_{p=1}^N \rho_{np} C_{pm} \sharp_{mm} \exp[i\bar{\mu}_m u_{om} + i\lambda_n v_{on} + i\bar{\mu}_m^2 t + i\lambda_n^2 t] \\
\tau_2^{mn} &= \eta_{mn} \exp[-i\mu_m u_{om} - i\lambda_n v_{on} - i\mu_m^2 t - i\lambda_n^2 t] \\
\tilde{\tau}_2^{mn} &= \sum_{p=1}^N \eta_{mp} D_{pn} \lambda_{nn} \exp[-i\mu_m u_{om} - i\bar{\lambda}_n v_{on} - i\mu_m^2 t - i\bar{\lambda}_n^2 t]
\end{aligned} \tag{3.89}$$

where the time dependence is explicitly given,

$$\begin{aligned}
\sharp_{mn} &= \bar{\mu}_m - \mu_n \\
\lambda_{mn} &= \bar{\lambda}_m - \lambda_n,
\end{aligned} \tag{3.90}$$

ρ and η are arbitrary complex constant matrices, v_{on} and u_{on} are arbitrary real constants. We shall show that the choice of the boundaries $a_0^{(1)}$ and $a_0^{(2)}$ determines uniquely the complex matrices C and D , together with the λ and the μ , and that the other parameters are left free. For definiteness we choose

$$\text{Im } \lambda_n < 0, \quad \text{Im } \mu_n > 0. \quad (3.91)$$

By computing the residua at the poles $k = \lambda_n$ and $k = \mu_n$ of both sides of the integral equations (3.56) we get

$$\begin{aligned} \text{Res}(\psi_1, \lambda_n) &= \begin{pmatrix} \text{Res}(\psi_{01}, \lambda_n) \\ 0 \end{pmatrix} + G_1^- \text{Res}(\psi_1, \lambda_n) \\ \text{Res}(\psi_2, \mu_n) &= \begin{pmatrix} 0 \\ \text{Res}(\psi_{02}, \mu_n) \end{pmatrix} + G_2^+ \text{Res}(\psi_2, \mu_n). \end{aligned} \quad (3.92)$$

By inserting these integral equations into the r.h.s. of equation (3.61) considered at the special values $k = \lambda_n$ and $k = \mu_n$ and by recalling that the $\bar{\partial}$ -derivative of a pole at $k = k_0$ is given by

$$\frac{\partial}{\partial \bar{k}} \frac{1}{k - k_0} = -2\pi i \delta(k - k_0) \quad (3.93)$$

and the symmetry property of the Green operator G

$$\begin{aligned} G_2^+ &= G_1^- \\ G_2^- &= G_1^- + \frac{1}{2} \begin{pmatrix} 0 & \int du q(u, v) \\ 0 & 0 \end{pmatrix} \\ G_1^+ &= G_2^+ + \frac{1}{2} \begin{pmatrix} 0 & 0 \\ \int dv r(u, v) & 0 \end{pmatrix} \end{aligned} \quad (3.94)$$

we get, for the special choice of $R_d(k, l)$ in (3.86),

$$\begin{aligned} \text{Res}(\psi_1, \lambda_n) &= \mathcal{F}_1 + G_1^- \text{Res}(\psi_1, \lambda_n) \\ \text{Res}(\psi_2, \mu_n) &= \mathcal{F}_2 + G_2^+ \text{Res}(\psi_2, \mu_n) \end{aligned} \quad (3.95)$$

where

$$\mathcal{F}_1 = \begin{pmatrix} \mathcal{F}_{11} \\ \mathcal{F}_{21} \end{pmatrix} \quad (3.96)$$

with

$$\begin{aligned} \mathcal{F}_{11} &= \frac{1}{4} \sum_{m=1}^N \int du q \psi_{22}^-(\bar{\mu}_m) (\rho C)_{nm} \#_{mm} \exp[i\bar{\mu}_m u_{om} + i\lambda_n v_{on} + i\bar{\mu}_m^2 t + i\lambda_n^2 t] \\ \mathcal{F}_{21} &= \frac{1}{2} \sum_{m=1}^N \rho_{nm} \{ \text{Res}(\psi_{02}^+, \mu_m) \exp[i\mu_m u_{om} + i\lambda_n v_{on} + i\mu_m^2 t + i\lambda_n^2 t] + \\ &\quad \sum_{p=1}^N C_{mp} \#_{pp} \psi_{02}^-(\bar{\mu}_p) \exp[i\bar{\mu}_p u_{op} + i\lambda_n v_{on} + i\bar{\mu}_p^2 t + i\lambda_n^2 t] \} \end{aligned} \quad (3.97)$$

and

$$\mathcal{F}_2 = \begin{pmatrix} \mathcal{F}_{12} \\ \mathcal{F}_{22} \end{pmatrix} \quad (3.98)$$

with

$$\begin{aligned} \mathcal{F}_{22} &= -\frac{1}{4} \sum_{m=1}^N \int dv r \psi_{11}^+(\bar{\lambda}_m) (\eta D)_{nm} \lambda_{mm} \exp[-i\mu_n u_{on} - i\bar{\lambda}_m v_{om} - i\mu_n^2 t - i\bar{\lambda}_m^2 t] \\ \mathcal{F}_{12} &= -\frac{1}{2} \sum_{m=1}^N \eta_{nm} \left\{ \text{Res}(\psi_{01}^-, \lambda_m) \exp[-i\mu_n u_{on} - i\lambda_m v_{om} - i\mu_n^2 t - i\lambda_m^2 t] + \right. \\ &\quad \left. \sum_{p=1}^N D_{mp} \lambda_{pp} \psi_{01}^+(\bar{\lambda}_p) \exp[-i\mu_n u_{on} - i\bar{\lambda}_p v_{op} - i\mu_n^2 t - i\bar{\lambda}_p^2 t] \right\} \end{aligned} \quad (3.99)$$

By comparing the integral equations (3.92) and (3.95) one gets

$$\begin{aligned} \text{Res}(\psi_{01}^-, \lambda_n) &= \\ \frac{1}{4} \sum_{m=1}^N \int du q \psi_{22}^-(\bar{\mu}_m) (\rho C)_{nm} \sharp_{mm} \exp[i\bar{\mu}_m u_{om} + i\lambda_n v_{on} + i\bar{\mu}_m^2 t + i\lambda_n^2 t] \end{aligned} \quad (3.100)$$

$$\begin{aligned} \text{Res}(\psi_{02}^+, \mu_n) &= \\ -\frac{1}{4} \sum_{m=1}^N \int dv r \psi_{11}^+(\bar{\lambda}_m) (\eta D)_{nm} \lambda_{mm} \exp[-i\mu_n u_{on} - i\bar{\lambda}_m v_{om} - i\mu_n^2 t - i\bar{\lambda}_m^2 t] \end{aligned} \quad (3.101)$$

and

$$\text{Res}(\psi_{01}^-, \lambda_n) = - \sum_{m=1}^N D_{nm} \lambda_{mm} \psi_{01}^+(\bar{\lambda}_m) \exp[i\lambda_n v_{on} - i\bar{\lambda}_m v_{om} + i\lambda_n^2 t - i\bar{\lambda}_m^2 t] \quad (3.102)$$

$$\text{Res}(\psi_{02}^+, \mu_n) = - \sum_{m=1}^N C_{nm} \sharp_{mm} \psi_{02}^-(\bar{\mu}_m) \exp[-i\mu_n u_{on} + i\bar{\mu}_m u_{om} - i\mu_n^2 t + i\bar{\mu}_m^2 t] \quad (3.103)$$

Equations (3.102) and (3.103) determine uniquely the boundary values $a_0^{(1)}$ and $a_0^{(2)}$ of the auxiliary field $A = \text{diag}(A^{(1)}, A^{(2)})$. In fact, they determine the Spectral Transform of $a_0^{(1)}$ and $a_0^{(2)}$ considered as potentials in the time dependent Schrödinger equations of (3.65). If we use the Spectral Transform defined in the previous section

$$\frac{\partial \psi_{01}}{\partial k} = \int \int dl \wedge d\bar{l} \psi_{01}(l) r_1(k, l) e^{i(k^2 - l^2)t} \quad (3.104)$$

$$\frac{\partial \psi_{02}}{\partial k} = \int \int dl \wedge d\bar{l} \psi_{02}(l) r_2(k, l) e^{-i(k^2 - l^2)t} \quad (3.105)$$

we get that

$$r_1(k, l) = 2\pi i \sum_{n,m=1}^N \exp[i\lambda_n v_{on} - i\bar{\lambda}_m v_{om}] D_{nm} \lambda_{mm} \delta(l - \bar{\lambda}_m) \delta(k - \lambda_n) \quad (3.106)$$

$$r_2(k, l) = 2\pi i \sum_{n,m=1}^N \exp[-i\mu_n u_{on} + i\bar{\mu}_m u_{om}] C_{nm} \sharp_{mm} \delta(l - \bar{\mu}_m) \delta(k - \mu_n) \quad (3.107)$$

It results, in particular, that ψ_{01} and ψ_{02} have simple poles at $k = \lambda_n$ and at $k = \mu_n$ as expected.

If we consider the KPI equation associated, for instance, to the time dependent Schrödinger equation for $a_0^{(1)}(v, t)$ by taking into account that (v, t) are to be considered as “space” variables of the KPI equation while its “time” variable has to be considered as an additional parameter of the potential $a_0^{(1)}(v, t)$, it results that the obtained boundary value $a_0^{(1)}(v, t)$ coincides with an N wave soliton solution of the KPI equation in the (v, t) plane at some fixed “time”. Analogously, $a_0^{(2)}(u, t)$ can be considered as an N wave soliton solution of a KPI equation in the (u, t) “plane” at some fixed “time”. The real parameters v_{on} and u_{on} fix the position of the n^{th} wave soliton in the corresponding “plane”.

If we require $a_0^{(1)}$ and $a_0^{(2)}$ to be real the complex matrices C and D satisfy

$$D_{nm}\lambda_{mm} = \overline{D_{mn}}\lambda_{nn} \quad (3.108)$$

$$C_{nm}\#_{mm} = \overline{C_{mn}}\#_{nn}. \quad (3.109)$$

Moreover, without any loss of generality, by eventually shifting u_{on} and v_{on} we can choose

$$D_{nn}^2 = C_{nn}^2 = 1 \quad (3.110)$$

Equations (3.100) and (3.101) solve the direct problem furnishing the matrices ρ and η in terms of the eigenfunction of the spectral problem (3.54), of Q and of the parameters and eigenfunctions defining the boundary. By using the orthogonality relations derived in the previous section

$$\begin{aligned} i \int dv \psi_{01}^+(\bar{\lambda}_m) \overline{\text{Res}(\psi_{01}^-, \lambda_n)} &= \delta_{mn} \\ i \int du \psi_{02}^-(\bar{\mu}_m) \overline{\text{Res}(\psi_{02}^+, \mu_n)} &= \delta_{mn} \end{aligned} \quad (3.111)$$

we obtain the formulae

$$\begin{aligned} \rho_{nm} &= -i \exp[-i\lambda_n v_{on} - i\lambda_n^2 t] (X^{-1})_{nm} \\ \eta_{nm} &= -i \exp[i\mu_n u_{on} + i\mu_n^2 t] (Y^{-1})_{nm} \end{aligned} \quad (3.112)$$

where

$$\begin{aligned} X_{ns} &= \frac{1}{4} \sum_{m=1}^N C_{nm}\#_{mm} e^{i\bar{\mu}_m u_{om} + i\bar{\mu}_m^2 t} \iint du dv q \psi_{22}^-(\bar{\mu}_m) \overline{\psi_{01}^+(\bar{\lambda}_s)} \\ Y_{ns} &= \frac{1}{4} \sum_{m=1}^N D_{nm}\lambda_{mm} e^{-i\bar{\lambda}_m v_{om} - i\bar{\lambda}_m^2 t} \iint du dv r \psi_{11}^+(\bar{\lambda}_m) \overline{\psi_{02}^-(\bar{\mu}_s)}. \end{aligned} \quad (3.113)$$

It is worth noting that the chosen parameterization of the spectral data in (3.89) allows us to discriminate between the parameters $\lambda_n, \mu_n, u_{on}, v_{on}, C_{mn}, D_{mn}$ which are fixed by the choice of the boundary values $a_0^{(1)}$ and $a_0^{(2)}$ and parameters ρ_{mn}, η_{mn} which are left free and are expected to govern the specific nonlinear dynamic of the DSI equation.

In analogy with the one dimensional case we call ρ_{mn} and η_{mn} the normalization matrix coefficients.

In the reduced case $r = \sigma_0 \bar{q}$ from equation (3.80) computed at $k = \mu_n$ and at $l = \lambda_m$, by using the orthogonality relations (3.111), we get the necessary condition

$$\sum_{m=1}^N \rho_{nm} C_{ms} \#_{ss} = \sigma_0 \sum_{m=1}^N D_{nm} \lambda_{mm} \bar{\eta}_{sm} \quad (3.114)$$

In solving the inverse problem we shall prove that this condition is also sufficient for having $r = \sigma_0 \bar{q}$.

We are left with the solution of the inverse problem, i.e. we have to reduce the reconstruction of the matrix Q and the auxiliary field A from given spectral data $R(k, l)$ to the solution of a linear problem.

The $\bar{\partial}$ -equation (3.61) together with the asymptotic requirement in (3.55) defines a non-local Riemann–Hilbert problem for the matrix function

$$\phi(x, y, t, k) = \psi(x, y, t, k) e^{-ik(\sigma_3 x - y)} \quad (3.115)$$

Its solution is obtained by solving the singular linear integral equation (the space-time variables of ϕ are understood)

$$\phi(k) = \mathbb{1} + \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{dh \wedge d\bar{h}}{h - k} \iint_{\mathbb{C}} dl \wedge d\bar{l} \phi(l) e^{il(\sigma_3 x - y)} R(h, l, t) e^{-ih(\sigma_3 x - y)} \quad (3.116)$$

This equation furnishes an asymptotic expansion in powers of $\frac{1}{k}$ of ϕ

$$\phi = \mathbb{1} + \frac{1}{k} \phi^{(1)} + O\left(\frac{1}{k^2}\right) \quad (3.117)$$

where

$$\phi^{(1)} = -\frac{1}{2\pi i} \iint_{\mathbb{C}} dh \wedge d\bar{h} \iint_{\mathbb{C}} dl \wedge d\bar{l} \phi(l) e^{il(\sigma_3 x - y)} R(h, l, t) e^{-ih(\sigma_3 x - y)}. \quad (3.118)$$

Then the solution of the inverse problem is achieved by inserting the expansion into the Zakharov–Shabat spectral problem $T_1 \psi = 0$ and into the auxiliary spectral problem $T_2 \psi = \psi \Omega$ and by identifying the coefficients of the powers of $\frac{1}{k}$. One obtains Q , A and a useful expression for Q^2

$$Q = i[\sigma_3, \phi^{(1)}] \quad (3.119)$$

$$A = -i(\partial_x - \sigma_3 \partial_y) \text{diag } \phi^{(1)} \quad (3.120)$$

$$Q^2 = 2i\sigma_3(\partial_x + \sigma_3 \partial_y) \text{diag } \phi^{(1)}. \quad (3.121)$$

When only the discrete part of the spectrum is present one can derive explicit algebraic formulae for Q and A . The requirement that ϕ has simple poles at $k = \lambda_n$ and at $k = \mu_n$ and that ϕ goes to $\mathbb{1}$ in the large k limit fixes a k dependence of the form

$$\phi = \mathbb{1} + \sum_{n=1}^N \left(\frac{\varphi_{n1} e^{\beta_n}}{k - \lambda_n}, \frac{\varphi_{n2} e^{\alpha_n}}{k - \mu_n} \right) \quad (3.122)$$

where, for convenience, in the residua an explicit exponential factor has been introduced with

$$\alpha_n = i\mu_n(u - u_{on}) - i\mu_n^2 t \quad (3.123)$$

$$\beta_n = -i\lambda_n(v - v_{on}) + i\lambda_n^2 t \quad (3.124)$$

The vectors φ_{n1} and φ_{n2} are computed by inserting (3.122) into the $\bar{\partial}$ -equation (3.61) with $R(k, l)$ from (3.86) and (3.89). One obtains

$$\left[\mathbb{1} + \frac{1}{4} \eta \mathcal{B} \rho \mathcal{A} \right] \varphi_2 = -\frac{1}{2} \eta \delta \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{4} \eta \mathcal{B} \rho \gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.125)$$

$$\left[\mathbb{1} + \frac{1}{4} \rho \mathcal{A} \eta \mathcal{B} \right] \varphi_1 = \frac{1}{2} \rho \gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{4} \rho \mathcal{A} \eta \delta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.126)$$

where

$$\gamma_n = \sum_{m=1}^N C_{nm} \sharp_{mm} e^{\bar{\alpha}_m} \quad (3.127)$$

$$\delta_n = \sum_{m=1}^N D_{nm} \lambda_{mm} e^{\bar{\beta}_m} \quad (3.128)$$

$$\mathcal{A} = \mathbb{1} + C\alpha, \quad \alpha_{nm} = \frac{\sharp_{nn}}{\sharp_{nm}} e^{\bar{\alpha}_n + \alpha_m} \quad (3.129)$$

$$\mathcal{B} = \mathbb{1} + D\beta, \quad \beta_{nm} = \frac{\lambda_{nn}}{\lambda_{nm}} e^{\bar{\beta}_n + \beta_m}. \quad (3.130)$$

From equation (3.119) one gets

$$Q = 2i\sigma_3 \sum_{n=1}^N \begin{pmatrix} 0 & (\varphi_1)_2 e^{\alpha_n} \\ (\varphi_2)_1 e^{\beta_n} & 0 \end{pmatrix} \quad (3.131)$$

and from (3.120) and (3.121) by the rule for differentiating a determinant one gets

$$A = 2 \begin{pmatrix} \partial_v^2 & 0 \\ 0 & -\partial_u^2 \end{pmatrix} \ln \Delta \quad (3.132)$$

$$Q^2 = -4\partial_u \partial_v \ln \Delta \quad (3.133)$$

where

$$\Delta = \det \left[\mathbb{1} + \frac{1}{4} \eta \mathcal{B} \rho \mathcal{A} \right] = \det \left[\mathbb{1} + \frac{1}{4} \rho \mathcal{A} \eta \mathcal{B} \right]. \quad (3.134)$$

By solving, in a similar way, the $\bar{\partial}$ -equations (3.104) and (3.105) one can derive also the boundaries

$$a_0^{(1)} = 2\partial_v^2 \ln \det \mathcal{B} \quad (3.135)$$

$$a_0^{(2)} = -2\partial_u^2 \ln \det \mathcal{A}. \quad (3.136)$$

In order to study the reduction and the regularity properties of the found solution it is convenient to introduce the hermitian matrices

$$\tilde{C}_{mn} = iC_{mn}\sharp_{nn} \quad (3.137)$$

$$\tilde{D}_{mn} = -iD_{mn}\lambda_{nn}. \quad (3.138)$$

One can easily get that

$$\tilde{C}\mathcal{A}^\dagger = \mathcal{A}\tilde{C} \quad (3.139)$$

$$\tilde{D}\mathcal{B}^\dagger = \mathcal{B}\tilde{D} \quad (3.140)$$

Moreover, if the constraint (3.114), which rewritten by using \tilde{C} and \tilde{D} reads

$$\tilde{D}\eta^\dagger = -\sigma_0\rho\tilde{C}, \quad (3.141)$$

is satisfied one can easily verify directly that the reduction condition

$$r = \sigma_0\bar{q} \quad (3.142)$$

is fulfilled. We conclude that the constraint (3.114) is necessary and sufficient for having the reduction.

To see that (3.125) and (3.126) can be solved for φ_1 and φ_2 and that, consequently, Q and A are regular, we first note that the two hermitian matrices

$$\tilde{\alpha}_{nm} = -i \frac{e^{\bar{\alpha}_n + \alpha_m}}{\sharp_{nm}} \quad (3.143)$$

$$\tilde{\beta}_{nm} = i \frac{e^{\bar{\beta}_n + \beta_m}}{\lambda_{nm}} \quad (3.144)$$

are positive definite. The corresponding quadratic forms in the dummy vector P can be written as

$$\sum_{n,m=1}^N \bar{P}_n P_m \tilde{\alpha}_{nm} = \sum_{n,m=1}^N \bar{P}_n P_m \int_u^{+\infty} du' e^{\bar{\alpha}_n + \alpha_m} = \int_u^{+\infty} du' \left| \sum_{n=1}^N P_n e^{\alpha_n} \right|^2 \quad (3.145)$$

$$\sum_{n,m=1}^N \bar{P}_n P_m \tilde{\beta}_{nm} = \sum_{n,m=1}^N \bar{P}_n P_m \int_{-\infty}^v dv' e^{\bar{\beta}_n + \beta_m} = \int_{-\infty}^v dv' \left| \sum_{n=1}^N P_n e^{\beta_n} \right|^2 \quad (3.146)$$

which are positive unless all P_n are zero. Therefore if the matrices \tilde{C} and \tilde{D} are chosen to be positive definite, because the product of two positive hermitian matrices is positive $\mathcal{A} = \mathbb{1} + C\alpha = \mathbb{1} + \tilde{C}\tilde{\alpha}$ and $\mathcal{B} = \mathbb{1} + D\beta = \mathbb{1} + \tilde{D}\tilde{\beta}$ are positive definite and the boundaries $a_0^{(1)}$ and $a_0^{(2)}$ are regular. If, moreover, the reduction condition (3.137) is satisfied the matrices (3.125) and (3.126) can be rewritten as

$$\mathbb{1} + \frac{1}{4}\eta\mathcal{B}\rho\mathcal{A} = \mathbb{1} - \frac{1}{4}\sigma_0\eta\mathcal{B}\tilde{D}\eta^\dagger\tilde{C}^{-1}\mathcal{A} \quad (3.147)$$

$$\mathbb{1} + \frac{1}{4}\rho\mathcal{A}\eta\mathcal{B} = \mathbb{1} - \frac{1}{4}\sigma_0\rho\mathcal{A}\tilde{C}\rho^\dagger\tilde{D}^{-1}\mathcal{B} \quad (3.148)$$

which for $\sigma_0 = -1$ are positive definite for regular η and ρ . For $\sigma_0 = 1$ they are positive definite for η and ρ matrices with sufficiently small norms.

3.3 The dynamics of localized solitons in two dimensions

The solution q , in the generic case, describes N^2 localized coherent structures interacting in a complicated way at finite times, but moving in the far past and in the far future with constant velocities $\mathbf{V}_{nm} = (2\text{Re } \lambda_n, 2\text{Re } \mu_m)$ and without changing form. It is therefore natural to call this solution the N^2 soliton solution. It is parameterized by a point in a space of $4N(N+1)$ real parameters. Of these parameters $2N(N+2)$ and, precisely, $\lambda_n, \mu_n, v_{on}, u_{on}, C$ and D are determined by the choice of the boundaries and fix the velocity and the possible location of the solitons in the plane, while the remaining $2N^2$ and, precisely, η and ρ govern the dynamics of the solitons during the interaction.

A relevant information in the global dynamical behaviour of the solitons is furnished by the mass (energy, charge, or number of particles according to the physical context) of the solution q

$$M = \iint |q|^2 du dv \quad (3.149)$$

and by the masses of the solitons at $t = \pm\infty$

$$M_{mn}^{(\pm)} = \iint |q_{mn}^{(\pm)}|^2 dudv \quad (m, n = 1, 2, \dots, N) \quad (3.150)$$

where $q_{mn}^{(\pm)}$ is the asymptotic behaviour of q at $t = \pm\infty$ computed in the rest reference frame of the (m, n) soliton. If $\det(\rho\eta) \neq 0, \det \alpha \neq 0$ and $\det \beta \neq 0$ the mass M is given by the simple formula

$$M = -4\sigma_0 \ln \frac{\det(\mathbb{1} + \frac{1}{4}\eta\rho)}{\det(\frac{1}{4}\eta\rho)}. \quad (3.151)$$

In general, it results that $M = \sum_{mn} M_{mn}^{(-)} = \sum_{mn} M_{mn}^{(+)}$. However, the mass of the single soliton is not conserved and in particular it can be zero at $t = +\infty$ or at $t = -\infty$. For a special choice of the parameters the mass of the (m, n) soliton can be zero at $t = \pm\infty$ also when the coefficients ρ_{mn} and η_{nm} are not both equal to zero. We call these solitons with zero mass virtual solitons and they generate peculiar effects as in figure 1 of [8] where a virtual soliton collides with a soliton forcing it to change velocity.

On the other hand, the total momentum of q

$$\mathbf{P} = (P_1, P_2), \quad P_1 = i \iint (\bar{q}_u q - \bar{q} q_u) du dv, \quad P_2 = i \iint (\bar{q}_v q - \bar{q} q_v) du dv \quad (3.152)$$

is not conserved and, in fact, it results that

$$\frac{d\mathbf{P}}{dt} = \left(2 \iint |q|^2 \frac{\partial}{\partial u} A_2(u, t) du dv, -2 \iint |q|^2 \frac{\partial}{\partial v} A_1(v, t) du dv \right) \quad (3.153)$$

where

$$A_1 = a_0^{(1)} - \frac{1}{4}\sigma_0 \int du (|q|^2)_v, \quad A_2 = a_0^{(2)} + \frac{1}{4}\sigma_0 \int dv (|q|^2)_u. \quad (3.154)$$

Because the four-soliton solution displays all the richness of the general case we examine it in detail. We choose for definiteness $\lambda_{n\Im} < 0, \mu_{n\Im} > 0$ ($n = 1, 2$), $(\lambda_{2\Re} -$

$\lambda_{1\Re} > 0$ and $(\mu_{2\Re} - \mu_{1\Re}) < 0$. It is convenient to parameterize the matrices ρ and η in such a way that the reduction condition (3.114) is automatically satisfied, i.e.

$$\eta = \frac{1}{d} \begin{pmatrix} (\bar{\ell}_1 - \bar{\mathcal{D}}m_1)\sqrt{\frac{\#_{11}}{\lambda_{11}}} & (m_1 - \mathcal{D}\bar{\ell}_1)\sqrt{\frac{\#_{11}}{\lambda_{22}}} \\ (\bar{m}_2 - \bar{\mathcal{D}}\ell_2)\sqrt{\frac{\#_{22}}{\lambda_{11}}} & (\ell_2 - \mathcal{D}\bar{m}_2)\sqrt{\frac{\#_{22}}{\lambda_{22}}} \end{pmatrix} \quad (3.155)$$

$$\rho = -\frac{\sigma_0}{c} \begin{pmatrix} (\ell_1 - \bar{\mathcal{C}}m_2)\sqrt{\frac{\lambda_{11}}{\#_{11}}} & (m_2 - \mathcal{C}\ell_1)\sqrt{\frac{\lambda_{11}}{\#_{22}}} \\ (\bar{m}_1 - \bar{\mathcal{C}}\ell_2)\sqrt{\frac{\lambda_{22}}{\#_{11}}} & (\bar{\ell}_2 - \mathcal{C}\bar{m}_1)\sqrt{\frac{\lambda_{22}}{\#_{22}}} \end{pmatrix} \quad (3.156)$$

with ℓ_n and m_n ($n = 1, 2$) arbitrary complex parameters and

$$\mathcal{C} = C_{12}\sqrt{\frac{\#_{22}}{\#_{11}}}, \quad \mathcal{D} = D_{12}\sqrt{\frac{\lambda_{22}}{\lambda_{11}}} \quad (3.157)$$

$$c = 1 - |\mathcal{C}|^2, \quad d = 1 - |\mathcal{D}|^2. \quad (3.158)$$

The parameters $\lambda_n, \mu_n, \mathcal{C}$ and \mathcal{D} fix the boundaries while ℓ_n and m_n govern the dynamics of the soliton interaction. In the case $\sigma_0 = -1$ if $c > 0$ and $d > 0$ the solution is regular.

The masses of the solitons can be explicitly computed. For a generic choice of the parameters they do not depend on the spectral parameters λ_m, μ_n and on the initial positions (u_{om}, v_{on}) . We get ($\sigma_0 = -1$)

$$M_{11}^{(-)} = 4 \ln \left(1 + 4 \frac{c |\ell_2 - \mathcal{D}\bar{m}_2|^2}{|\bar{\ell}_1\ell_2 - m_1\bar{m}_2|^2 + 4cd |m_2|^2} \right) \quad (3.159)$$

$$M_{22}^{(-)} = 4 \ln \left(1 + 4 \frac{d |\ell_1 - \bar{\mathcal{C}}m_2|^2}{|\bar{\ell}_1\ell_2 - m_1\bar{m}_2|^2 + 4cd |m_2|^2} \right) \quad (3.160)$$

$$M_{11}^{(+)} = 4 \ln \left(1 + 4 \frac{d |\ell_2 - \bar{\mathcal{C}}m_1|^2}{|\bar{\ell}_1\ell_2 - m_1\bar{m}_2|^2 + 4cd |m_1|^2} \right) \quad (3.161)$$

$$M_{22}^{(+)} = 4 \ln \left(1 + 4 \frac{c |\ell_1 - \mathcal{D}\bar{m}_1|^2}{|\bar{\ell}_1\ell_2 - m_1\bar{m}_2|^2 + 4cd |m_1|^2} \right) \quad (3.162)$$

$$M_{12}^{(-)} = 4 \ln \left(1 + 4 \frac{|4dc\bar{m}_2 + (\ell_2\bar{\ell}_1 - \bar{m}_2m_1)[(\ell_1 - \bar{\mathcal{C}}m_2)\bar{\mathcal{D}} - (\bar{m}_1 - \bar{\mathcal{C}}\ell_2)]|^2}{[|\bar{\ell}_1\ell_2 - m_1\bar{m}_2|^2 + 4cd(|m_2 - \bar{\ell}_2\mathcal{D}|^2 + d|m_2|^2)]} \times \frac{1}{[|\bar{\ell}_1\ell_2 - m_1\bar{m}_2|^2 + 4d(|m_2 - \ell_1\mathcal{C}|^2 + c|m_1|^2)]} \right) \quad (3.163)$$

$$M_{21}^{(-)} = 4 \ln \left(1 + 4dc \frac{|m_2|^2}{|\bar{\ell}_1\ell_2 - m_1\bar{m}_2|^2} \right) \quad (3.164)$$

$$M_{12}^{(+)} = 4 \ln \left(1 + 4dc \frac{|m_1|^2}{|\bar{\ell}_1\ell_2 - m_1\bar{m}_2|^2} \right) \quad (3.165)$$

$$M_{21}^{(+)} = 4 \ln \left(1 + 4 \frac{|4dc\bar{m}_1 + (\ell_2\bar{\ell}_1 - \bar{m}_2m_1)[(\bar{\ell}_2 - \mathcal{C}\bar{m}_1)\mathcal{D} - (m_2 - \mathcal{C}\ell_1)]|^2}{[|\bar{\ell}_1\ell_2 - m_1\bar{m}_2|^2 + 4cd(|m_1 - \bar{\ell}_1\mathcal{D}|^2 + d|m_1|^2)]} \times \frac{1}{[|\bar{\ell}_1\ell_2 - m_1\bar{m}_2|^2 + 4d(|m_1 - \ell_2\mathcal{C}|^2 + c|m_2|^2)]} \right) \quad (3.166)$$

Of special interest are the cases in which one or more masses are zero. For definiteness, we choose the masses $M_{mn}^{(\pm)}$ ($m \neq n$) different from zero and we consider the case in which some $M_{mm}^{(\pm)}$ are zero.

For $\ell_1 = \bar{\mathcal{C}}m_2$ or $\ell_2 = \mathcal{D}\bar{m}_2$ one gets $M_{22}^{(-)} = 0$ or $M_{11}^{(-)} = 0$ and the solution describes the creation of a soliton. For $\ell_1 = \mathcal{D}\bar{m}_1$ or $\ell_2 = \bar{\mathcal{C}}m_1$ one gets $M_{22}^{(+)} = 0$ or $M_{11}^{(+)} = 0$, i.e. the annihilation of a soliton. For $\ell_1 = \bar{\mathcal{C}}m_2$ and $\ell_2 = \mathcal{D}\bar{m}_2$ it results that $M_{22}^{(-)} = M_{11}^{(-)} = 0$ and the solution describes the creation of a pair of solitons. For $\ell_1 = \mathcal{D}\bar{m}_1$ and $\ell_2 = \bar{\mathcal{C}}m_1$ one gets $M_{22}^{(+)} = M_{11}^{(+)} = 0$, i.e. the annihilation of a pair of solitons. For $\ell_1 = \bar{\mathcal{C}}m_2$ and $\ell_2 = \bar{\mathcal{C}}m_1$ one gets $M_{22}^{(-)} = M_{11}^{(+)} = 0$, i.e. a soliton changes its mass and velocity. For $\ell_1 = \mathcal{D}\bar{m}_1$, $\ell_2 = (|\mathcal{D}|^2/\mathcal{C})m_1$, $m_2 = (\mathcal{D}/\bar{\mathcal{C}})\bar{m}_1$ one gets $M_{22}^{(\pm)} = M_{11}^{(-)} = 0$ and the solution describes the creation of a soliton. For $\ell_1 = \mathcal{D}\bar{m}_1$, $\ell_2 = (|\mathcal{C}|^2/\bar{\mathcal{D}})\bar{m}_2$, $m_2 = (\mathcal{D}/\bar{\mathcal{C}})\bar{m}_1$ one gets $M_{22}^{(\pm)} = M_{11}^{(+)} = 0$ and the solution describes the annihilation of a soliton. For $\ell_1 = \ell_2 = \bar{\mathcal{C}}m$, $m_1 = m_2 = m$, $\bar{\mathcal{C}}m = \mathcal{D}\bar{m}$ all the masses $M_{mm}^{(\pm)}$ are zero and one gets a solution describing two interacting solitons.

All these dynamical behaviours can be obtained by choosing the same boundaries. Boundaries do not give any information on the dynamics of solitons, but fix only their kinematics, i.e. their possible locations in the plane and their velocities.

Of special interest is the case in which two spectral data are equal, say $\mu_1 = \mu_2$. In general, this solution describes the interaction of two solitons. The masses $M_{mm}^{(\pm)}$ ($m = 1, 2$) in this degenerate case depend also on the initial position of the solitons. With a special choice of the parameters one can get a solution describing the fission of a soliton (see figure 2 in [8]) and the fusion of two solitons. If in addition one chooses $\det(\rho\eta) = 0$ the solution describes a single soliton that by the interaction with a virtual soliton is forced to change velocity (see figure 1 in [8]). Note that the two dynamical processes described in figure 1 and figure 2 in [8] are obtained by choosing the same boundaries and different matrices ρ and η .

3.4 Asymptotic bifurcation of multidimensional solitons

The degeneracy of the solution when two discrete spectral data are chosen to be equal is worth of a deeper analysis.

This phenomenon is well known in one dimension, but, while in one dimension by taking two eigenvalues equal in the N -soliton solution we recover the $(N-1)$ -soliton solution, in 2+1 dimensions, as shown by the previous example, we get a new solution.

To understand the underlying mechanism it is convenient to consider the simplest case in which the effect takes place. Precisely, we choose $N = 2$, the matrices ρ and η diagonal

$$\rho = \text{diag}(\rho_1, \rho_2), \quad \eta = \text{diag}(\eta_1, \eta_2) \quad (3.167)$$

and

$$D = \mathbb{1}, \quad C = \mathbb{1}. \quad (3.168)$$

In the reduced case $\sigma_0 = -1$ (we are considering) the two matrices η and ρ are related by the constraint

$$\rho_n \#_{nn} = -\bar{\eta}_n \lambda_{nn}. \quad (3.169)$$

The set of these solutions describes a family of geometrical objects evolving in time in the (x, y) plane; each object $E_{(p,t)}$ of the family is parameterized by a point p in a P

space of 16 real parameters and by the time and therefore corresponds to a point in a 17-dimensional space $S = \{p, t\}$. We call generically stable or generic those objects E_{s_0} of the family that depend in a differentiable way on the 17 parameters in a neighborhood of a point s_0 . According to the usual definition in catastrophe theory we call the complement of this open set $\{s_0\}$ the set of bifurcation points.

The generic solution describes two solitons mutually interacting without changing shape and velocity. The only effect of the interaction is a shift in the position and in the overall phase.

The only bifurcation points are $(p, -\infty)$ and $(p, +\infty)$ with a special choice of the parameters p . Precisely, when any couple of discrete eigenvalues λ_n, λ_m or μ_n, μ_m have the same real part, i.e. when the parameters belong to the hyperplanes $\lambda_{n\Re} = \lambda_{m\Re}$ or $\mu_{n\Re} = \mu_{m\Re}$ in P , the two-soliton solution is not stable at large times. For this special choice of the parameters one gets solitons that, as a result of their mutual interaction, exhibit a two dimensional shift and also a change of form. If we require, in addition, to the representative point p of the two-soliton solution to belong to the hyperplanes of lower dimensions $\lambda_n = \lambda_m$ or $\mu_n = \mu_m$ the two solitons, because of their mutual interaction, not only are shifted in the plane and change their form but also exchange mass. Surprisingly enough, in both cases the relevance of the bifurcation effect depends on the relative initial position of the two solitons.

In order to describe a generic soliton with parameters λ, μ and $\gamma = \frac{1}{4}\eta\rho$ we introduce, in agreement with the notation for the one soliton solution, the two variables

$$\begin{aligned}\xi_1 &= -\mu_{\Im}(u - u_0) - \lambda_{\Im}(v - v_0) + 2(\lambda_{\Im}\lambda_{\Re} + \mu_{\Im}\mu_{\Re})t \\ \xi_2 &= \mu_{\Im}(u - u_0) - \lambda_{\Im}(v - v_0) + 2(\lambda_{\Im}\lambda_{\Re} - \mu_{\Im}\mu_{\Re})t\end{aligned}\quad (3.170)$$

the phase

$$\phi = \mu_{\Re}(u - u_0) + \lambda_{\Re}(v - v_0) + (\lambda_{\Im}^2 - \lambda_{\Re}^2 + \mu_{\Im}^2 - \mu_{\Re}^2)t \quad (3.171)$$

and the functions

$$\begin{aligned}a^{(\pm)} &= \gamma \exp(\pm \xi_1), \quad b^{(+)} = (1 + \gamma) \exp(\xi_2), \quad b^{(-)} = \gamma \exp(-\xi_2) \\ D(\xi_1, \xi_2) &= a^{(+)} + a^{(-)} + b^{(+)} + b^{(-)}.\end{aligned}\quad (3.172)$$

To any of these variables and functions we add a label (n) when we are using the parameters λ_n, μ_n and $\gamma_n = \frac{1}{4}\eta_n\rho_n$ of the n th soliton.

Then the two-soliton solution can be written as

$$q = -2i \frac{N}{\Delta} \quad (3.173)$$

where

$$\begin{aligned}N &= \frac{1}{2}\eta_1\lambda_{11} \exp(i\phi_{(1)}) \left[|\lambda_{12}|^2 |\mu_{12}|^2 b_{(2)}^{(+)} + \mu_{12}\mu_{12}\lambda_{21}\bar{\lambda}_{21}b_{(2)}^{(-)} + \right. \\ &\quad \left. |\lambda_{12}|^2 \mu_{12}\mu_{12}a_{(2)}^{(+)} + |\mu_{12}|^2 \lambda_{21}\bar{\lambda}_{21}a_{(2)}^{(-)} \right] + (1 \leftrightarrow 2) \\ \Delta &= |\lambda_{12}|^2 |\mu_{12}|^2 b_{(1)}^{(-)}b_{(2)}^{(-)} + |\lambda_{12}|^2 |\mu_{12}|^2 \left[b_{(1)}^{(-)}a_{(2)}^{(-)} + a_{(1)}^{(-)}b_{(2)}^{(-)} + a_{(1)}^{(-)}a_{(2)}^{(-)} \right] + \\ &\quad |\lambda_{12}|^2 |\mu_{12}|^2 \left[b_{(1)}^{(-)}a_{(2)}^{(+)} + a_{(1)}^{(+)}b_{(2)}^{(-)} + a_{(1)}^{(+)}a_{(2)}^{(+)} \right] +\end{aligned}\quad (3.174)$$

$$\begin{aligned}
& |\lambda_{12}|^2 |\#_{12}|^2 \left[b_{(1)}^{(+)} b_{(2)}^{(+)} + a_{(1)}^{(+)} b_{(2)}^{(+)} + b_{(1)}^{(+)} a_{(2)}^{(+)} + a_{(1)}^{(-)} b_{(2)}^{(+)} + \right. \\
& \left. b_{(1)}^{(+)} a_{(2)}^{(-)} + b_{(1)}^{(-)} b_{(2)}^{(+)} + b_{(1)}^{(+)} b_{(2)}^{(-)} + a_{(1)}^{(+)} a_{(2)}^{(-)} + a_{(1)}^{(-)} a_{(2)}^{(+)} \right] - \\
& \frac{1}{2} \text{Re} \left[\eta_1 \bar{\eta}_2 \lambda_{11} \lambda_{22} \lambda_{21} \#_{12} \exp(i(\phi_{(1)} - \phi_{(2)})) \right]
\end{aligned} \tag{3.175}$$

with

$$\mu_{12} = \mu_1 - \mu_2, \quad \lambda_{12} = \lambda_1 - \lambda_2. \tag{3.176}$$

We consider the discrete values λ_n fixed, with for instance $\lambda_{2\Re} > \lambda_{1\Re}$, and we study the bifurcations of the solution at large time with respect to the parameters μ_n . There is a bifurcation at $\mu_{1\Re} = \mu_{2\Re}$ and at $\mu_1 = \mu_2$. We need therefore to compute separately the asymptotic behaviour of the two-soliton solution in the two cases. In particular we study the asymptotic behaviour $q_{(2)}^{(\mp)}(u, v)$ of the second soliton at $t = \mp\infty$. Thanks to the symmetry of the two-soliton solution the asymptotic behaviour of the other soliton is simply obtained by exchanging in the formulae the labels 1 and 2.

For $\mu_{1\Re} = \mu_{2\Re}$ we get

$$\begin{aligned}
q_{(2)}^{(-)} &= -2\eta_2 \lambda_{2\Im} \exp[i(\phi_{(2)} - \phi_{(2)}^{(-)})] \frac{1 + \exp(-i\phi_0^{(2)}) E_{(2)}}{D_{(2)} + E_{(2)} \tilde{D}_{(2)}} \left(\xi_1^{(2)} - \xi_0^{(2)(-)}, \xi_2^{(2)} - \xi_0^{(2)(-)} \right) \\
q_{(2)}^{(+)} &= -2\eta_2 \lambda_{2\Im} \exp[i\phi_{(2)}] \frac{1 + \frac{\gamma_1}{1+\gamma_1} \exp(-i\phi_0^{(2)}) E_{(2)}}{D_{(2)} + \frac{\gamma_1}{1+\gamma_1} E_{(2)} \tilde{D}_{(2)}} \left(\xi_1^{(2)}, \xi_2^{(2)} \right)
\end{aligned} \tag{3.177}$$

with

$$\begin{aligned}
E_{(2)} \left(\xi_1^{(2)}, \xi_2^{(2)} \right) &= \exp \left[\frac{\mu_{1\Im}}{\mu_{2\Im}} \left(\xi_1^{(2)} - \xi_2^{(2)} \right) + \xi_0^{(2)} \right] c_{12} \\
c_{12} &= \exp[2\mu_{1\Im}(u_{01} - u_{02})] \\
\tilde{D}_{(2)} \left(\xi_1^{(2)}, \xi_2^{(2)} \right) &= D_{(2)} \left(\xi_1^{(2)} + \xi_0^{(2)}, \xi_2^{(2)} - \xi_0^{(2)} \right) \\
\xi_0^{(2)} &= \ln \left| \frac{\mu_{12}}{\#_{12}} \right|, \quad \phi_0^{(2)} = \arg \bar{\mu}_{12} \#_{12} \\
\xi_0^{(2)(-)} &= \ln \left| \frac{\lambda_{12}}{\lambda_{12}} \right|, \quad \phi_{(2)}^{(-)} = \arg \bar{\lambda}_{12} \lambda_{12}.
\end{aligned} \tag{3.178}$$

The mass of the second soliton at $t = \pm\infty$ is

$$M_{(2)}^{(\pm)} = 4 \ln \frac{1 + \gamma_2}{\gamma_2}. \tag{3.179}$$

Therefore the position and the phase of the soliton are shifted, its shape is changed but it does not exchange mass with the other soliton. Note that its shape depends on the relative initial position of the two solitons.

For $\mu_1 = \mu_2$ we get

$$\begin{aligned}
q_{(2)}^{(-)} &= \frac{-2\eta_2 \lambda_{2\Im} \exp[i(\phi_{(2)} - \phi_{(2)}^{(-)})]}{\left[\left(1 + c_{12} \frac{1+\gamma_2}{\gamma_2} \right) a_{(2)}^{(+)} + a_{(2)}^{(-)} + b_{(2)}^{(+)} + (1 + c_{12}) b_{(2)}^{(-)} \right] \left(\xi_1^{(2)} - \xi_0^{(2)(-)}, \xi_2^{(2)} - \xi_0^{(2)(-)} \right)} \\
q_{(2)}^{(+)} &= \frac{-2\eta_2 \lambda_{2\Im} \exp[i\phi_{(2)}]}{\left[\left(1 + c_{12} \frac{\gamma_1}{\gamma_2} \frac{1+\gamma_2}{1+\gamma_1} \right) a_{(2)}^{(+)} + a_{(2)}^{(-)} + b_{(2)}^{(+)} + \left(1 + c_{12} \frac{1+\gamma_1}{\gamma_1} \right) b_{(2)}^{(-)} \right] \left(\xi_1^{(2)}, \xi_2^{(2)} \right)}
\end{aligned} \tag{3.180}$$

and

$$\begin{aligned} M_{(2)}^{(-)} &= 4 \ln \frac{(1 + \gamma_2)(1 + c_{21})}{c_{12} + \gamma_2(1 + c_{21})} \\ M_{(2)}^{(+)} &= 4 \ln \frac{(1 + \gamma_2)(1 + \gamma_1 + c_{21}\gamma_1)}{\gamma_2(1 + \gamma_1) + c_{21}\gamma_1(1 + \gamma_2)}. \end{aligned} \quad (3.181)$$

Therefore the soliton during the interaction shifts its position, changes shape and exchanges mass with the other soliton, while the total mass of the two solitons is conserved. It is worth noting that the shape and the energy of each soliton depend on the relative initial position of the two solitons.

Drawings describing the peculiar behaviours of solitons described in this section can be found in [6].

Finally let us remark that also the boundary $a^{(2)}(u, t)$ bifurcates. In fact, while in the generic case $a^{(2)}(u, t)$ describes in the (u, t) plane two infinite waves crossing at one point, for $\mu_{1\Re} = \mu_{2\Re}$ it describes two parallel infinite waves and for $\mu_1 = \mu_2$ only one infinite wave.

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